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TOPICS IN SCATTERING THEORY AND VENEZIANO MODEL

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled TOPICS IN SCATTERING THEORY AND VENEZIANO MODEL, submitted by Nan-Nyoh Wong in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

This thesis is divided into two parts.

Part I

In the usual N/D calculations for the partial waves in potential scattering only the left hand singularities nearest the threshold are taken into account. This approximation does not yield a partial wave amplitude (for $\ell \geq 1$) satisfying the threshold condition that the partial wave amplitudes behave like s^{ℓ} near threshold. In this part of the thesis an additional pole is introduced to simulate the effects of the distant singularities. The parameters of this pole are determined by imposing the threshold condition. Both s and p wave amplitudes for scattering by an exponential potential are calculated.

Part II

The Veneziano model, which does not satisfy unitarity, has had remarkable successes in predicting the low energy behaviour of many scattering processes, like $\pi\pi$, πK and KK scatterings. In this part of the thesis various aspects of the Veneziano model are investigated.

The s and p partial wave projection of the Veneziano amplitude has been made for $\pi\pi$, πK and $K\bar{K}$ (and KK) scatterings and the scattering lengths have been calculated.

The $\pi\pi \rightarrow \pi A_1$ reaction has been discussed with an alternative way of imposing the Adler zero condition on the amplitudes.

Finally, the $K_2^0 \rightarrow \pi^+\pi^-\gamma$ and $K_2^0 \rightarrow \gamma\gamma$ decays are considered in the pole model, in which the intermediate state is taken as an off-mass shell pion rather than an on-shell pion and the strong interaction part described by the Veneziano model.

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PART I

THRESHOLD PROBLEM IN N/D METHOD FOR PARTIAL WAVES IN POTENTIAL SCATTERING

Introduction

The N/D formalism(1) has been extensively used to calculate the partial wave amplitude satisfying unitarity. If the discontinuity across the entire left hand cut for the partial wave were known one could, via the N/D formalism, generate the exact partial wave amplitude. An exact solution would also satisfy the correct threshold condition, namely,

$$f_\ell(\sqrt{s}) \sim (s)^\ell \quad \text{as} \quad s \to 0 ,$$

where $\sqrt{s}$ is the centre of mass three momentum. In practice one does not know the entire left hand singularity and one hopes that the effect of the singularities nearest the threshold would be large and thus the calculation of the N function (and therefore of $f_\ell(s)$) proceeds with the neglect of the distant singularities. This type of approximation does not always yield a partial wave amplitude which satisfies the threshold condition. The vanishing of the partial wave amplitude for $\ell \geq 1$ at threshold can be looked upon as a result of delicate cancellations between the contributions of the left hand cut and the right hand cut. Any approximation to the left hand cut will upset this delicate balance. One way to enforce a correct threshold behaviour is to work with an amplitude(2)(3)

$$\tilde{f}_\ell(s) = f_\ell(s)/(s)^\ell .$$

This however leads to unpleasant high energy behaviour and a cut-off is required(3) in such calculations to keep integrals finite.

An improvement in the usual N/D calculation can be achieved, as suggested by Dilley(4), by adding to the nearby poles a pole on the left hand side to simulate the effects of the distant singularities. The parameters of this pole can be determined by imposing the threshold conditions. Based on this idea a calculation has been presented in the following chapter for the scattering of spinless bosons by an exponential potential whose s wave phase shift has been discussed in detail by some authors(5)(6). The scattering problem has been solved exactly for $\ell = 0$ and $\ell = 1$ and the exact result has then be compared with the various approximation schemes. For an exponential potential the exact solution for $\ell = 0$ can be written down analytically(7)(8). Even though for s-wave there is no threshold problem as such since f_ℓ goes to a constant, we have evaluated this constant in various approximation schemes.

The s wave problem is treated in Section I and the p wave problem in Section II of Chapter I. The exact phase shift for p-wave was obtained by a numerical method suggested by Calogero(9).

CHAPTER I

N/D METHOD FOR SCATTERING BY AN EXPONENTIAL POTENTIAL

We shall study the scattering of two identical spinless particles of mass m by an exponential potential,

$$U(r) = -\lambda\beta^2 e^{-\beta r} \quad . \tag{1.1}$$

Here λ is a dimensionless constant and β has dimension of mass. $U(r)$ of (1.1) is the usual potential $V(r)$ multiplied by $m/\hbar^2$. The radial Schroedinger equation takes the form

$$\frac{d^2u(r)}{dr^2} + \left[s - U(r) - \frac{\ell(\ell+1)}{r^2}\right] u(r) = 0 \ , \tag{1.2}$$

where $s = k^2 = mE/\hbar^2$. The total scattering amplitude is defined in terms of the partial wave amplitudes as

$$f(s, \cos\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)P_\ell(\cos\theta)f_\ell(s) \ , \tag{1.3}$$

where

$$f_\ell(s) = \frac{1}{\sqrt{s}}\, e^{i\delta_\ell(s)} \sin\delta_\ell(s) = \frac{1}{\sqrt{s}\,\cot\delta_\ell(s) - i\sqrt{s}} \quad . \tag{1.4}$$

Section I : S Wave Solution

(i) Exact Solution

With an exponential potential the Schroedinger equation is exactly soluble for s waves and the solution is given by (10)

$$u_o(r) = \phi_o^{(-)}(s)e^{i\sqrt{s}r} + \phi_o^{(+)}(s)e^{-i\sqrt{s}r} . \qquad (1.5)$$

The coefficient $\phi_o^{(-)}$ and $\phi_o^{(+)}$ are as follows

$$\phi_o^{(\mp)}(s) = [J_{\pm 2i\sqrt{s}/\beta}(2\sqrt{\lambda})]\sqrt{\lambda}^{\pm 2i\sqrt{s}/\beta}\Gamma(1 \pm 2i \frac{\sqrt{s}}{\beta}) , \qquad (1.6)$$

where $J_{2i\sqrt{s}/\beta}(2\sqrt{\lambda})$ is the Bessel's function of order $2i\sqrt{s}/\beta$, and $\Gamma(1 \pm 2i \frac{\sqrt{s}}{\beta})$ is the gamma-function.

The s wave amplitude is then given by

$$f_o(s) = \frac{1}{2i\sqrt{s}} \frac{\phi_o^{(-)}(s) - \phi_o^{(+)}(s)}{\phi_o^{(+)}(s)} . \qquad (1.7)$$

By expanding the ϕ functions in powers of λ and s and retaining only the first two terms in each expansion we find the phase shift given by

$$\sqrt{s}\cot\delta_o(s) = \mathrm{Re}[f_o(s)]^{-1} = \frac{\beta}{2\lambda}[1 - \frac{5}{8}\lambda + 0(\lambda^2)]$$

$$+ \frac{s}{\beta\lambda}[2 + \frac{9}{16}\lambda + 0(\lambda^2)] + 0(s^2) . \qquad (1.8)$$

Equation (1.8) gives the exact expansion for cot $\delta_o(s)$ for small λ to order λ^o and to first order in s. In the remaining part of this section, we shall solve the problem in various approximation schemes and then compare the results with that of equation (1.8).

If the asymptotic limit ($s \to \infty$) of the exact solution is taken, one finds that

$$\sqrt{s}\ \cot\delta(s) \xrightarrow[s\to\infty]{} \frac{\beta\lambda}{2}\,\frac{1}{s + (\beta^2/4)} \tag{1.9}$$

which we shall see is the first Born approximation.

(ii) First Born Approximation

The first Born approximation to the s-wave amplitude is given by

$$f_{1B}(s) = -\frac{1}{s}\int_0^\infty \sin^2(\sqrt{s}r)U(r)dr = \frac{\beta\lambda}{2}\,\frac{1}{s + (\beta^2/4)} \quad . \tag{1.10}$$

Let us write this as

$$f_{1B}(s) = \Gamma_1/(s + \alpha_1^2) \ , \tag{1.11}$$

with $\Gamma_1 = \beta\lambda/2$ and $\alpha_1^2 = \beta^2/4$. The first Born approximation gives a simple pole on the negative real axis. The phase shift is given by

$$\sqrt{s}\ \cot\delta_o(s) = [f_{1B}(s)]^{-1} = \frac{\beta}{2\lambda} + \frac{2s}{\beta\lambda} \quad . \tag{1.12}$$

Comparing this with (1.8), we see that it reproduces the order λ^{-1} term correctly. To produce terms of order λ^o we have to go to the next term in Born series.

(iii) Second Born Approximation

In the second Born approximation the s-wave amplitude is given by

$$f_{2B}(s) = f_{1B}(s) + \frac{1}{s} \iint \sin(\sqrt{s}r)U(r)G(r,r')U(r')\sin(\sqrt{s}r')dr'dr \tag{1.13}$$

where $G(r,r')$ is the Green function,

$$G(r,r') = \frac{\sin(\sqrt{s}r')\cos(\sqrt{s}r)}{\sqrt{s}} \qquad \text{for } r' < r$$

$$= \frac{\sin(\sqrt{s}r)\cos(\sqrt{s}r')}{\sqrt{s}} \qquad \text{for } r' > r \quad .$$

After carrying out the relevant integrations in (1.13), we get

$$\sqrt{s}\cot\delta_o(s) = [f_{2B}(s)]^{-1}$$

$$= \frac{\beta}{2\lambda}(1 - \frac{5}{8}\lambda) + \frac{s}{\beta\lambda}(2 + \frac{9}{16}\lambda) \tag{1.14}$$

for small s. To order λ^o and to order s, this solution coincides with the exact solution (1.8).

(iv) Determinantal N/D Method

The N/D method in general consists in writing

$$f_\ell(s) = \frac{N_\ell(s)}{D_\ell(s)} \tag{1.15}$$

where $N_\ell(s)$ is assumed to be analytic in the s-plane apart from singularities on the negative real axis. D function on the other hand carries the right hand unitarity cut. Thus$^{(1)}$

$$N_\ell(s) = \frac{1}{\pi} \int_{-\infty}^{s_L} \frac{ds' D_\ell(s') \mathrm{Im}\, f_\ell(s')}{s' - s} \tag{1.16}$$

and

$$D_\ell(s) = 1 - \frac{1}{\pi} \int_0^\infty \frac{ds' \sqrt{s'}\, N_\ell(s')}{s' - s} \ . \tag{1.17}$$

The determinantal approximation$^{(11)}$ consists in assuming that the N function be approximated by the first Born term; thus,

$$N_o(s) = f_{1B}(s) \tag{1.18}$$

$$D_o(s) = 1 - \frac{1}{\pi} \int_0^\infty ds' \frac{\sqrt{s'}\, f_{1B}(s')}{s' - s} \ . \tag{1.19}$$

This approximation has the advantage of simplicity. It also ensures the correct threshold behaviour for $\ell \geq 1$ partial waves since the Born term satisfies the threshold condition. In a situation where an exact solution of the problem is difficult to obtain one hopes that such an approximation which amounts in a sense to the inclusion of only the largest range forces gives a reasonable description at low energies.

Substituting (1.10) in (1.19) one gets

$$D_o(s) = 1 - \frac{\Gamma_1}{\alpha_1 - i\sqrt{s}} \qquad (1.20)$$

and

$$f_o(s) = \left[\frac{\alpha_1(\alpha_1 - \Gamma_1) + s}{\Gamma_1} - i\sqrt{s}\right]^{-1} . \qquad (1.21)$$

The phase shift is given by (using parameters λ and β in favour of Γ_1 and α_1)

$$\sqrt{s}\cot\delta_o(s) = \frac{\beta}{2\lambda}(1 - \lambda) + \frac{2s}{\beta\lambda} . \qquad (1.22)$$

We notice that the order λ^{-1} terms have been reproduced correctly but not the order λ^{o} terms.

(v) One-pole N/D Method

The exact solution$^{(8)}$ of the s-wave problem shows that we have infinite number of poles on the left hand side. A correct and exact description of the scattering process in N/D type of calculation must take all these poles into account. Let us first make the simplest possible approximation and take only the nearest pole into account, then

$$\text{Im } f_o(s) = -\pi\Gamma_1\delta(s + \alpha_1^2) \qquad \text{for} \quad s < 0 . \qquad (1.23)$$

From (1.16) and (1.17) one gets

$$N_o(s) = \frac{\Gamma_1 D_o(-\alpha_1^2)}{s + \alpha_1^2} = R_1/(s + \alpha_1^2) \qquad (1.24)$$

where

$$R_1 = \Gamma_1 D_o(-\alpha_1^2) \tag{1.25}$$

and

$$D_o(s) = 1 - \frac{R_1}{\alpha_1 - i\sqrt{s}} \quad . \tag{1.26}$$

It is clear from (1.24) that the one-pole approximation normalizes $f_o(s)$ to $f_{1B}(s)$ at $s = \alpha_1^2$ whereas the determinantal method normalizes $f_o(s)$ to $f_{1B}(s)$ at $s = \infty$, we then get

$$f_o(s) = \left[\frac{\alpha_1^2 - \alpha_1 R_1 + s}{R_1} - i\sqrt{s}\right]^{-1} , \tag{1.27}$$

and the phase shift is given by

$$\sqrt{s}\ \cot\delta_o(s) = \frac{\beta}{2\lambda}\left(1 - \frac{\lambda}{2}\right) + \frac{2s}{\beta\lambda}\left(1 + \frac{\lambda}{2}\right) . \tag{1.28}$$

The leading terms are again correctly reproduced but order λ^o terms are incorrect and also different from that of the determinantal approximation.

(vi) Two-pole N/D Approximation

To simulate the effect of all but the nearest pole we shall add another pole to the left of the nearest pole. Thus one writes

$$\text{Im}\ f_o(s) = -\pi\Gamma_1\delta(s + \alpha_1^2) - \pi\Gamma_2\delta(s + \alpha_2^2) \quad \text{for} \quad s < 0 \tag{1.29}$$

with $\Gamma_1 = \beta\lambda/2$ and $\alpha_1^2 = \beta^2/4$. The parameters of the second pole could be determined if the two boundary conditions could be imposed on the partial wave. For partial waves $\ell \geq 1$ the threshold condition would be an obvious one to impose. For s-wave no threshold condition exists; however, we may demand that $f_o(s) \to f_{1B}(s)$ as $s \to \infty$. Thus only one (the residue) of these two new parameters can be determined. We have varied the other parameter (the position of the pole) and looked at the solution as a function of this parameter. On using (1.29) we get

$$N_o(s) = \frac{R_1}{s + \alpha_1^2} + \frac{R_2}{s + \alpha_2^2} \tag{1.30}$$

$$D_o(s) = 1 - \frac{R_1}{\alpha_1 - i\sqrt{s}} - \frac{R_2}{\alpha_2 - i\sqrt{s}} \tag{1.31}$$

with

$$R_i = \Gamma_i \, D(-\alpha_i) \qquad i=1,2 \, . \tag{1.32}$$

If now we demand the boundary condition that

$$f_o(s) \xrightarrow[s\to\infty]{} \Gamma_1/(s + \alpha_1^2) \; , \tag{1.33}$$

we find that

$$R_2 = R_1[1 - D(-\alpha_1^2)]/D(-\alpha_1^2) \tag{1.34}$$

or

$$R_2[1 - \frac{R_1}{2\alpha_1} - \frac{R_2}{\alpha_1 + \alpha_2}] = R_1[\frac{R_1}{2\alpha_1} + \frac{R_2}{\alpha_1 + \alpha_2}] \; . \tag{1.35}$$

From (1.35) and (1.33) we obtain the following

$$R_1 = \frac{2\Gamma_1\alpha_1 - \dfrac{2\alpha_1\,\Gamma_1^2}{\alpha_1 + \alpha_2}}{\Gamma_1 + 2\alpha_1 - \dfrac{2\alpha_1\Gamma_1}{\alpha_1+\alpha_2}} \tag{1.36}$$

$$R_2 = \frac{\Gamma_1^2}{\Gamma_1 + 2\alpha_1 - \dfrac{2\alpha_1\Gamma_1}{\alpha_1+\alpha_2}} \ . \tag{1.37}$$

Since Γ_1 is proportional to λ, R_2 starts off like λ^2 as indeed it should. The phase shift is then given by

$$\begin{aligned}\sqrt{s}\,\cot\delta_o(s) &= \frac{\mathrm{Re}\,D_o(s)}{N_o(s)} \\ &= \frac{s^2+s(\alpha_1^2+\alpha_2^2-R_1\alpha_1-R_2\alpha_2)+\alpha_1^2\alpha_2^2-R_1\alpha_1\alpha_2^2-R_2\alpha_2\alpha_1^2}{R_1(s+\alpha_2^2) + R_2(s+\alpha_1^2)} \ .\end{aligned} \tag{1.38}$$

Expanding in power of s and λ we get

$$\begin{aligned}\sqrt{s}\,\cot\delta_o(s) &= \frac{\beta}{2\lambda}\left[1 - \left(\frac{1}{2} + \frac{\beta^2}{8\alpha_2^2}\right)\lambda\right] \\ &+ \frac{2s}{\lambda}\left[1 + \left(\frac{1}{2} - \frac{\beta^2}{4\alpha_2^2} + \frac{\beta^4}{32\alpha_2^4}\right)\lambda\right] \ .\end{aligned} \tag{1.39}$$

We summarize our results in the following:

Parameters: $\lambda = 0.1$ and $\beta = 1$ in arbitrary units.

(i) Exact: $\sqrt{s}\,\cot\delta_o(s) = (4.687+0(\lambda))+(20.56+0(\lambda))s$

(ii) First Born: $= (5.00+0(\lambda))+(20.00+0(\lambda))s$

(iii) Second Born: $= (4.687+0(\lambda))+(20.56+0(\lambda))s$

(iv) N/D (Determinantal) $= (4.5+0(\lambda))+(20.00+0(\lambda))s$

(v) N/D (1-pole) $= (4.75+0(\lambda))+(21.00+0(\lambda))s$

(vi) N/D (2-pole) $= (4.5+0(\lambda))+(20.00+0(\lambda))s \quad (\alpha_2=\tfrac{1}{2})$

$= (4.687+0(\lambda))+(20.56+0(\lambda))s \quad (\alpha_2=1)$

$= (4.747+0(\lambda))+(20.98+0(\lambda))s \quad (\alpha_2=5)$

$= (4.749+0(\lambda))+(20.99+0(\lambda))s \quad (\alpha_2=10)$

$= (4.75+0(\lambda))+(21.00+0(\lambda))s \quad (\alpha_2\to\infty)$

It is clear that up to and including order λ^o terms, the second Born approximation and the 2-pole N/D method, with $\alpha_2=1$, give the exact solution. This is only to be expected since the second Born pole is located at $\alpha_2=1$. In the two-pole approximation if the second pole is superimposed on the first pole then the solution is the same as that in the determinantal approximation. This result again can be understood once it is realized that in the two-pole approximation we did impose the condition that $f_o(s)\xrightarrow[s\to\infty]{} f_{1B}(s)$ which is automatically satisfied by the determinantal method.

Section II : P Wave Solution

(i) Exact Solution

The Schroedinger equation for the p wave scattering from exponential potential cannot be solved analytically.

We adopt the method, given by Calogero$^{(9)}$, of a numerical solution of the integral equation

$$\tan\delta_\ell(s,r) = -\frac{1}{\sqrt{s}}\int_0^r U(r')[\hat{j}_\ell(\sqrt{s}r') - \tan\delta_\ell(s,r')\hat{n}_\ell(\sqrt{s}r')]^2 dr' \tag{1.40}$$

where the Riccati-Bessel functions $\hat{j}_\ell(\sqrt{s}r)$ and $\hat{n}_\ell(\sqrt{s}r)$ are defined as follows:

$$\hat{j}_\ell(\sqrt{s}r) = \sqrt{s}rj_\ell(\sqrt{s}r)$$

$$\hat{n}_\ell(\sqrt{s}r) = \sqrt{s}rj_\ell(\sqrt{s}r) \quad ,$$

in which $j_\ell(\sqrt{s}r)$ and $n_\ell(\sqrt{s}r)$ are the spherical Bessel functions of the first and second kind.

The function $\tan\delta_\ell(s,r)$ satisfies the boundary condition

$$\tan\delta_\ell(s,0) = 0 \quad \text{and} \quad \tan\delta_\ell(s,r\to\infty) = \tan\delta_\ell(s)$$

where $\tan\delta_\ell(s)$ is the tangent of the phase shift. Therefore, if the value of r in (1.40) is taken to be large enough in the numerical integration, we obtain the value for $\tan\delta_\ell$.

(ii) The Born Approximation

When the first Born term is calculated, we obtain the following

$$f_{1B}(s) = \frac{\lambda\beta}{2} [s + \frac{\beta^2}{4}]^{-1} (\frac{2s+\beta^2}{2s}) - \frac{\lambda\beta^3}{4s^2} \ln(1 + \frac{4s}{\beta^2}) \tag{1.41}$$

which has a pole at $s = -\beta^2/4$ and a cut runs from $s = -\beta^2/4$ to $-\infty$. It seems that we have also a pole at the origin, but when we expand both the terms in (1.41) in powers of s, the terms like $\frac{1}{s}$ are cancelled. As $s \to 0$, we also find that

$$f_{1B}(s=0) = 0 \tag{1.42}$$

$$f'_{1B}(s=0) = 8\lambda/3\beta^3 \tag{1.43}$$

and $$f''_{1B}(s=0) = -32\lambda/\beta^5 \quad . \tag{1.44}$$

The appearance of the cut in the Born term makes it quite inconvenient to solve for the p-wave amplitude using the N/D determinantal method in which the N function is taken as the first Born term. Thus, to simplify the calculation in the N/D approximation schemes, we shall make a further approximation of the first Born expression. We shall simulate the cut by a pole and write the Born expression in term of two poles, in which the first pole is located at $s = -\beta^2/4 = -s_{1B}$ and the second pole at $s = -s_{2B}$ whose value is yet to be determined

$$f_{1B}(s) = \frac{\Gamma_{1B}}{s + s_{1B}} + \frac{\Gamma_{2B}}{s + s_{2B}} \quad , \tag{1.45}$$

where Γ_{1B} and Γ_{2B} are the residues. Imposing the conditions (1.42)-(1.44), we get the following set of parameters

$$\Gamma_{1B} = -\frac{1}{3}\beta\lambda \,, \qquad \Gamma_{2B} = \frac{2}{3}\beta\lambda \quad \text{and} \quad s_{2B} = \frac{\beta^2}{2} \,. \tag{1.46}$$

Then, the Born term, which we expect to be well-approximated at low energies, becomes

$$f_{1B}(s) = -\frac{\beta\lambda}{3s + \frac{3}{4}\beta^2} + \frac{2\beta\lambda}{3s + \frac{3}{2}\beta^2} \,. \tag{1.47}$$

The phase shift obtained from the approximate first Born term is given by

$$s^{3/2}\cot\delta_1 = \frac{1}{\beta\lambda}\left[3s^2 + \frac{9}{4}\beta^2 s + \frac{3}{8}\beta^4\right] . \tag{1.48}$$

(iii) Determinantal N/D Method

We take the N function to be the approximate Born expression, (1.48), in which the left hand cut has been replaced by a second pole,

$$N_1(s) = \frac{(-\frac{1}{3})\beta\lambda}{s + (\frac{\beta^2}{4})} + \frac{(\frac{2}{3})\beta\lambda}{s + \frac{\beta^2}{2}} \,. \tag{1.49}$$

When substituting (1.49) into (1.17), we get the expression for D function;

$$D_1(s) = 1 + \frac{\beta\lambda/3}{\frac{\beta}{2} - i\sqrt{s}} - \frac{2\beta\lambda/3}{\sqrt{2}\,\beta - i\sqrt{s}} \,. \tag{1.50}$$

The phase shift is then given by

$$s^{3/2}\cot\delta_1 = \frac{1}{\beta\lambda}\left[3s^2 + \left(\frac{9}{4} + \frac{1-2\sqrt{2}}{2}\lambda\right)\beta^2 s + \left(\frac{3}{8} + \frac{1-\sqrt{2}}{4}\lambda\right)\beta^4\right] . \tag{1.51}$$

(iv) Three Pole N/D Approximation

Here, we approximate the left hand singularities in terms of three poles

$$\mathrm{Im}\, f_1(s) = -\pi\Gamma_1\delta(s+\alpha_1^2) - \pi\Gamma_2(s+\alpha_2^2) - \pi\Gamma_3\delta(s+\alpha_3^2) \tag{1.52}$$

where the first two poles are the two Born poles, that is, $\alpha_1^2=\beta^2/4$, $\alpha_2^2 = \frac{\beta^2}{2}$ with $\Gamma_1 = -\frac{1}{3}\beta\lambda$ and $\Gamma_2 = \frac{2}{3}\beta\lambda$, and the third pole is added to enforce the threshold condition. The N and D functions take the form,

$$N_1(s) = \frac{R_1}{s+\alpha_1^2} + \frac{R_2}{s+\alpha_2^2} + \frac{R_3}{s+\alpha_3^2} \tag{1.53}$$

$$D_1(s) = 1 - \frac{R_1}{\alpha_1 - i\sqrt{s}} - \frac{R_2}{\alpha_2 - i\sqrt{s}} - \frac{R_3}{\alpha_3 - i\sqrt{s}} \tag{1.54}$$

where the R functions are given by

$$R_i = \Gamma_i\, D(-\alpha_i^2) \qquad \text{for} \qquad i=1,2,3 . \tag{1.55}$$

There are two unknown parameters, α_3 and Γ_3, in the above equations. In principle two conditions would determine these parameters. However as we are working with an

approximate form for the Born term we can only impose the threshold boundary condition,

$$R_1\alpha_2^2\alpha_3^2 + R_2\alpha_1^2\alpha_3^2 + R_3\alpha_1^2\alpha_2^2 = 0 \ , \tag{1.56}$$

with confidence. The second condition that one could impose would be the asymptotic boundary condition which we, in our low energy approximation scheme, cannot enforce. However, we can introduce an asymptotic parameter as follows. From the exact Born expression (1.41), we see that

$$f^{(s)}_{\text{Born}} \longrightarrow \frac{\text{const}}{s} \qquad \text{as} \quad s \to \infty \ . \tag{1.57}$$

We may tentatively set

$$f_1(s) = \frac{N_1(s)}{D_1(s)} \xrightarrow[s\to\infty]{} \frac{R_1+R_2+R_3}{s} = \frac{c}{s} \quad \left(\text{or} = \frac{c'\Gamma_1}{s}\right) \tag{1.58}$$

where c is a free parameter and proportional to $\lambda\beta$ and $c' = \frac{c}{\Gamma_1}$ would be a pure number.

By solving the eqs. (1.55) and (1.58), we obtain

$$\left.\begin{aligned}
R_1 &\approx \Gamma_1 + \Gamma_1^2\left(\frac{1-c'}{\alpha_1+\alpha_3} - \frac{1}{2\alpha_1}\right) + \Gamma_1\Gamma_2\left(\frac{1}{\alpha_1+\alpha_3} - \frac{1}{\alpha_1+\alpha_2}\right) + 0(\lambda^3) \\
R_2 &\approx \Gamma_2 + \Gamma_2^2\left(\frac{1}{\alpha_2+\alpha_3} - \frac{1}{2\alpha_2}\right) + \Gamma_1\Gamma_2\left(\frac{1-c'}{\alpha_2+\alpha_3} - \frac{1}{\alpha_1+\alpha_2}\right) + 0(\lambda^3) \\
R_3 &\approx (c'-1)\Gamma_1 - \Gamma_2 + \Gamma_1^2\left(\frac{1}{2\alpha_1} - \frac{1-c'}{\alpha_1+\alpha_3}\right) + \Gamma_2^2\left(\frac{1}{2\alpha_2} - \frac{1}{\alpha_2+\alpha_3}\right) \\
&\quad + \Gamma_1\Gamma_2\left(\frac{2}{\alpha_1+\alpha_2} - \frac{1-c'}{\alpha_2+\alpha_3} - \frac{1}{\alpha_1+\alpha_3}\right) + 0(\lambda^3) \ .
\end{aligned}\right\} \tag{1.59}$$

Imposing the threshold condition (1.56) on (1.59), we get the expression for α_3^2,

$$\alpha_3^2 \left[\frac{\Gamma_1^2\alpha_2^2}{2\alpha_1} + \frac{\Gamma_1\Gamma_2(\alpha_1^2+\alpha_2^2)}{\alpha_1+\alpha_2} + \frac{\alpha_1^2\Gamma_2^2}{2\alpha_2}\right] = [\{(c'-1)\Gamma_1-\Gamma_2\}\alpha_1^2\alpha_2^2 -$$

$$\Gamma_1^2\left(\frac{1-2c'}{2}\right)\alpha_1\alpha_2^2 - \Gamma_1\Gamma_2\alpha_2^2\alpha_1 + \Gamma_1\Gamma_2\frac{\alpha_1^2\alpha_2^2}{\alpha_1+\alpha_2} -$$

$$(1-c')\alpha_1^2\alpha_2\Gamma_1\Gamma_2 - \frac{\Gamma_2^2\alpha_1^2\alpha_2}{2}] . \tag{1.60}$$

After putting the values $\Gamma_1 = -\beta\lambda/3$, $\alpha_1^2 = \beta^2/4$, $\Gamma_2 = 2\beta\lambda/3$ and $\alpha_2^2 = \beta^2/2$ in (1.60), we have

$$\alpha_3^2 = \frac{1}{4}[3(c'+1)(5\sqrt{2}+7)\frac{1}{\lambda} + 3 + \sqrt{2} + (6+4\sqrt{2})c']\beta^2. \tag{1.61}$$

One can now look for the probable values for c'.

1) If $f_\ell(s)$ approaches $f_{app.B}(s) = \beta\lambda/3$ in the limit $s \to \infty$, then from (1.58) we see that $c' = -1$ and

$$\alpha_3^2 = -\frac{3}{4}(1+\sqrt{2})\beta^2 .$$

This is not a desirable result as it places the pole in the unitarity continuum.

2) If we set $f_\ell(s) = f_{exact\ B}(s)$ as $s \to \infty$, then

$$R_1 + R_2 + R_3 = \beta\lambda/2 \qquad \text{or} \qquad c' = -3/2 .$$

α_3^2 now takes the value,

$$\alpha_3^2 = -\frac{1}{4}\left[\frac{3}{2\lambda}(5\sqrt{2}+7) - (6+5\sqrt{2})\right]\beta^2 .$$

Thus α_3^2 is positive definite only for negative values of λ. This is not a desirable result.

3) We are not happy with the way that α_3^2 depends on λ, when c' is required to take different set of values for positive and negative λ. If we were to solve the problem exactly (numerically) then there would, in principle, be no difficulty of this kind. The difficulty arises when we try to solve the problem approximately with some parameters determined from the threshold boundary condition and then try to impose an asymptotic condition on this approximate solution. Let us now define b via,

$$(c'+1) = b\lambda \quad \text{or} \quad c' = b\lambda - 1 , \tag{1.62}$$

then

$$\alpha_3^2 = \frac{\beta^2}{4}\left[3(5\sqrt{2}+7)b - 3(1+\sqrt{2}) + (6+4\sqrt{2})b\lambda\right] . \tag{1.63}$$

If λ is small, α_3^2 would not be effected by λ drastically. Dropping the third term in (1.63), we have

$$\alpha_3^2 = \frac{3}{4}\beta^2[5\sqrt{2}+7)b - (1+\sqrt{2})] , \tag{1.64}$$

which is greater than zero for $b > 3-2\sqrt{2} > 0$.

If we write $(c'-1)\Gamma_1 - \Gamma_2 = \Gamma_3$, then from (1.62) and the values of Γ_1 and Γ_2, one obtains

$$\Gamma_3 = -b\beta\lambda^2/3 \quad \text{which is of order } \lambda^2 .$$

The phase shift for the three pole N/D method is given by

$$s^{3/2}\cot\delta_1 = \{(R_1+R_2+R_3)s + [R_1(\alpha_2^2+\alpha_3^2) + R_2(\alpha_1^2+\alpha_3^2) + R_3(\alpha_1^2+\alpha_2^2)]\}^{-1} \times$$

$$\times \{s^3+s^2[\alpha_1^2+\alpha_2^2+\alpha_3^2-R_1\alpha_1-R_2\alpha_2-R_3\alpha_3]$$

$$+ s[\alpha_1^2\alpha_2^2+\alpha_2^2\alpha_3^2+\alpha_1^2\alpha_3^2-R(\alpha_2^2+\alpha_3^2)\alpha_1-R_2\alpha_2(\alpha_1^2+\alpha_3^2)-R_3\alpha_3(\alpha_1^2+\alpha_2^2)]$$

$$+ [\alpha_1^2\alpha_2^2\alpha_3^2 - R_1\alpha_1\alpha_2^2\alpha_3^2 - R_2\alpha_2\alpha_1^2\alpha_3^2 - R_3\alpha_3\alpha_1^2\alpha_2^2]\} . \qquad (1.65)$$

If we take $\alpha_3^2 = \frac{3}{4}\beta^2$ (a choice made with the expectation that $\alpha_3^2>\alpha_2^2$), then $\Gamma_3 = -(3\sqrt{2}-4)\beta\lambda^2/3$. We find from (1.59) that

$$R_1 = -\frac{1}{3}\beta\lambda + \frac{1}{9}\beta\lambda^2[4\sqrt{2}-5]$$

$$R_2 = \frac{2}{3}\beta\lambda + \frac{1}{9}\beta\lambda^2[2\sqrt{2}-4] \qquad (1.66)$$

$$R_3 = -\frac{1}{3}(5\sqrt{2}-7)\beta\lambda^2 .$$

Substituting (1.66) into (1.65), we have

$$s^{3/2}\cot\delta_1 = \{s\beta\lambda[0.333-0.0808\lambda] + \beta^3\lambda[0.25-0.056\lambda]\}^{-1} \times$$

$$\times \{s^3 + s\beta^2[1.5-0.3047\lambda+0.092\lambda^2]$$

$$+ s\beta^4[0.6875-0.2630\lambda+0.0796\lambda^2]$$

$$+ \beta^6[0.09062-0.0259\lambda+0.0060\lambda^2]\} \quad . \qquad (1.67)$$

The results for the p wave calculations are displayed in the plots of $s^{3/2}\cot\delta_1$ versus the energy s for $\lambda = \pm 0.1$ and $\lambda = \pm 0.5$ for $\beta = 1$. For negative λ the three pole N/D method gives better results than approximate Born solutions and determinantal results over a wide range of energy. For positive value of λ, the three pole N/D method is better in the region where s is small. We conclude then that the imposition of the threshold condition by putting an extra pole on the left hand real axis improves the result at low energies, in the sense that it is closer to the exact solution. If the solution could be obtained without making any of the approximations we have made, it would be expected that the improvement in the solution will extend over a much wider range of energy.

for $\lambda = -0.1$ $\beta = 1$

	Approx. Born	2 pole N/D	3 pole N/D	Exact
s = 0.1	-6.3	-6.4949	-6.247	-6.3391
s = 0.2	-9.4	-9.7363	-9.430	-9.4061
s = 0.3	-13.15	-13.5777	-13.19	-12.9628
s = 0.4	-17.55	-18.0191	-17.57	-16.9981
s = 0.5	-22.50	-23.0505	-22.48	-21.5033
s = 0.6	-28.05	-28.8019	-27.99	-26.4715
s = 0.7	-34.20	-34.9433	-34.1	-31.8974
s = 0.8	-40.95	-41.7847	-40.79	-37.7769
s = 0.9	-48.30	-49.2261	-48.08	-44.1064
s = 1.0	-56.25	-57.2675	-55.91	-50.8818

for $\lambda = -0.5$ $\beta = 1$

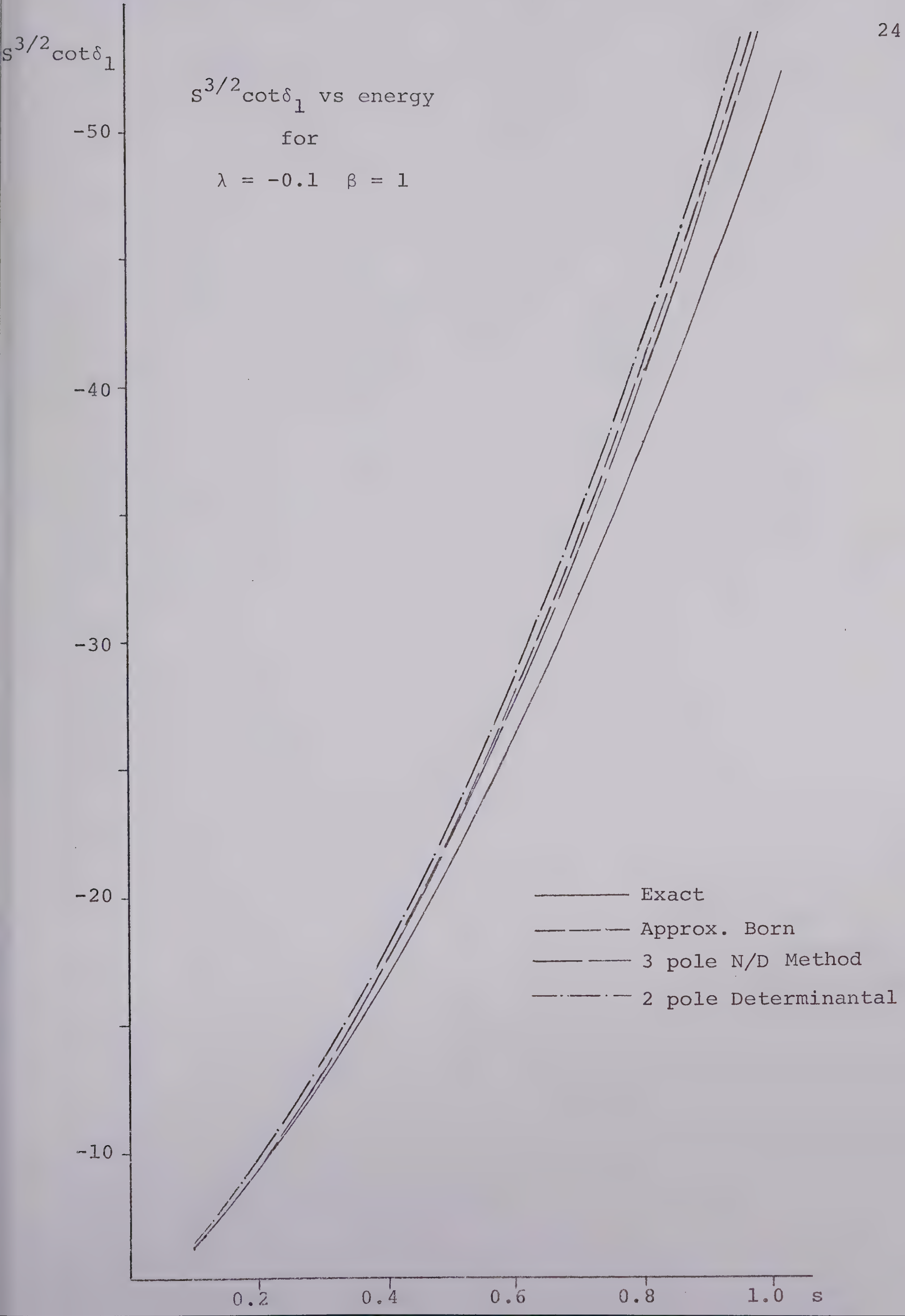
s = 0.1	-1.26	-1.4549	-1.319	-1.3247
s = 0.2	-1.88	-2.17634	-1.979	-1.9684
s = 0.3	-2.63	-3.0177	-2.746	-2.7054
s = 0.4	-3.51	-3.9791	-3.611	-3.5352
s = 0.5	-4.5	-5.0605	-4.599	-4.4546
s = 0.6	-5.61	-6.2619	-5.685	-5.4639
s = 0.7	-6.84	-9.5833	-6.955	-6.5617
s = 0.8	-8.19	-9.0247	-8.181	-7.7479
s = 0.9	-9.66	-10.5861	-9.588	-9.0221
s = 1.0	-11.25	-12.2677	-11.10	-10.3837

for $\lambda = 0.1$ $\beta = 1$

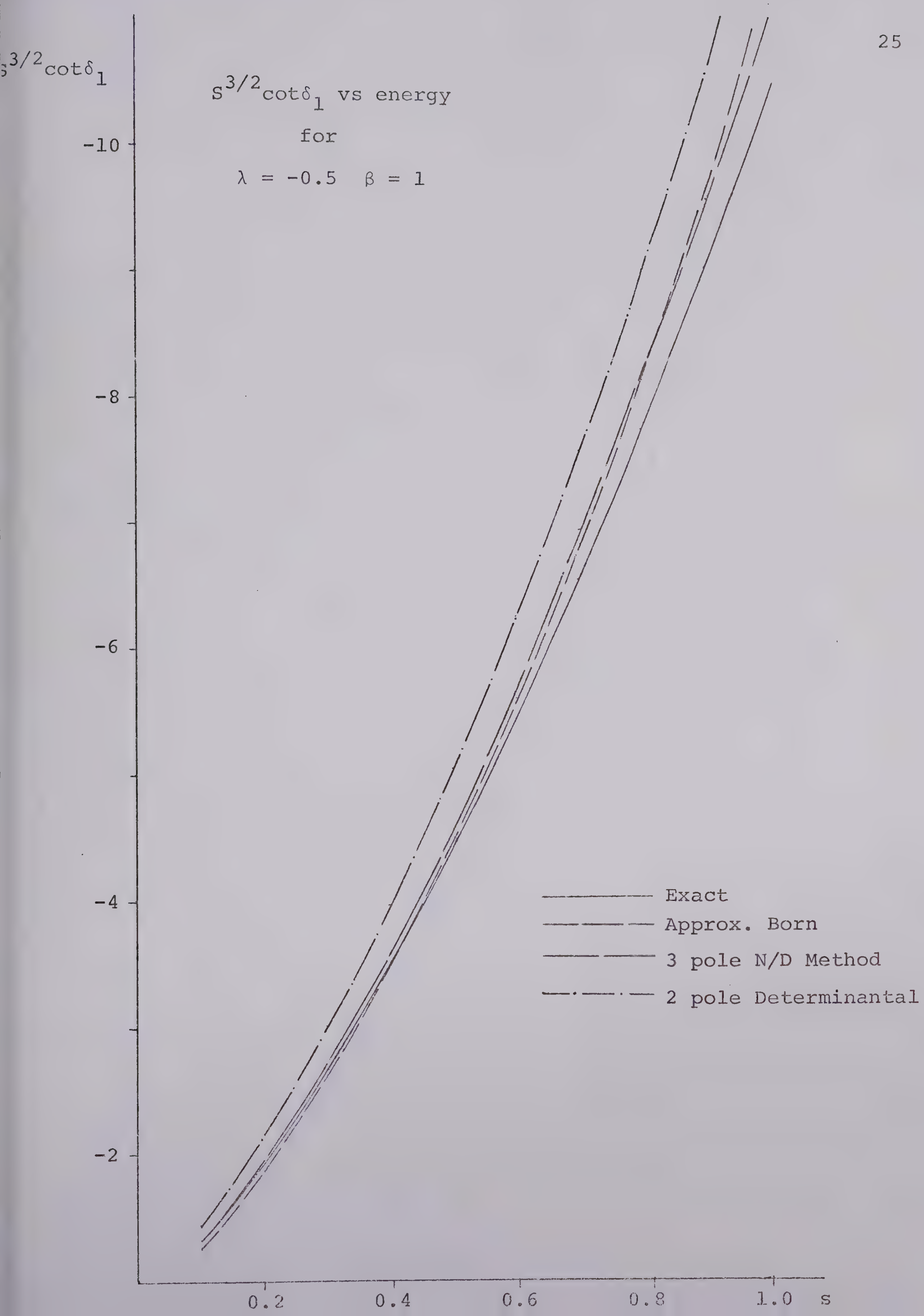
	Approx. Born	2 pole N/D	3 pole N/D	Exact
$s = 0.1$	6.3	6.1050	6.139	6.1929
$s = 0.2$	9.4	9.1636	9.29	9.1811
$s = 0.3$	13.15	12.8222	13.07	12.6697
$s = 0.4$	17.55	17.0808	17.42	16.6458
$s = 0.5$	22.50	21.8889	22.41	21.0994
$s = 0.6$	28.05	27.3979	27.99	26.0228
$s = 0.7$	34.20	33.4567	34.24	31.4098
$s = 0.8$	40.95	40.1151	41.05	37.2554
$s = 0.9$	48.30	47.373	48.50	43.555
$s = 1.0$	56.25	55.2323	56.54	50.3054

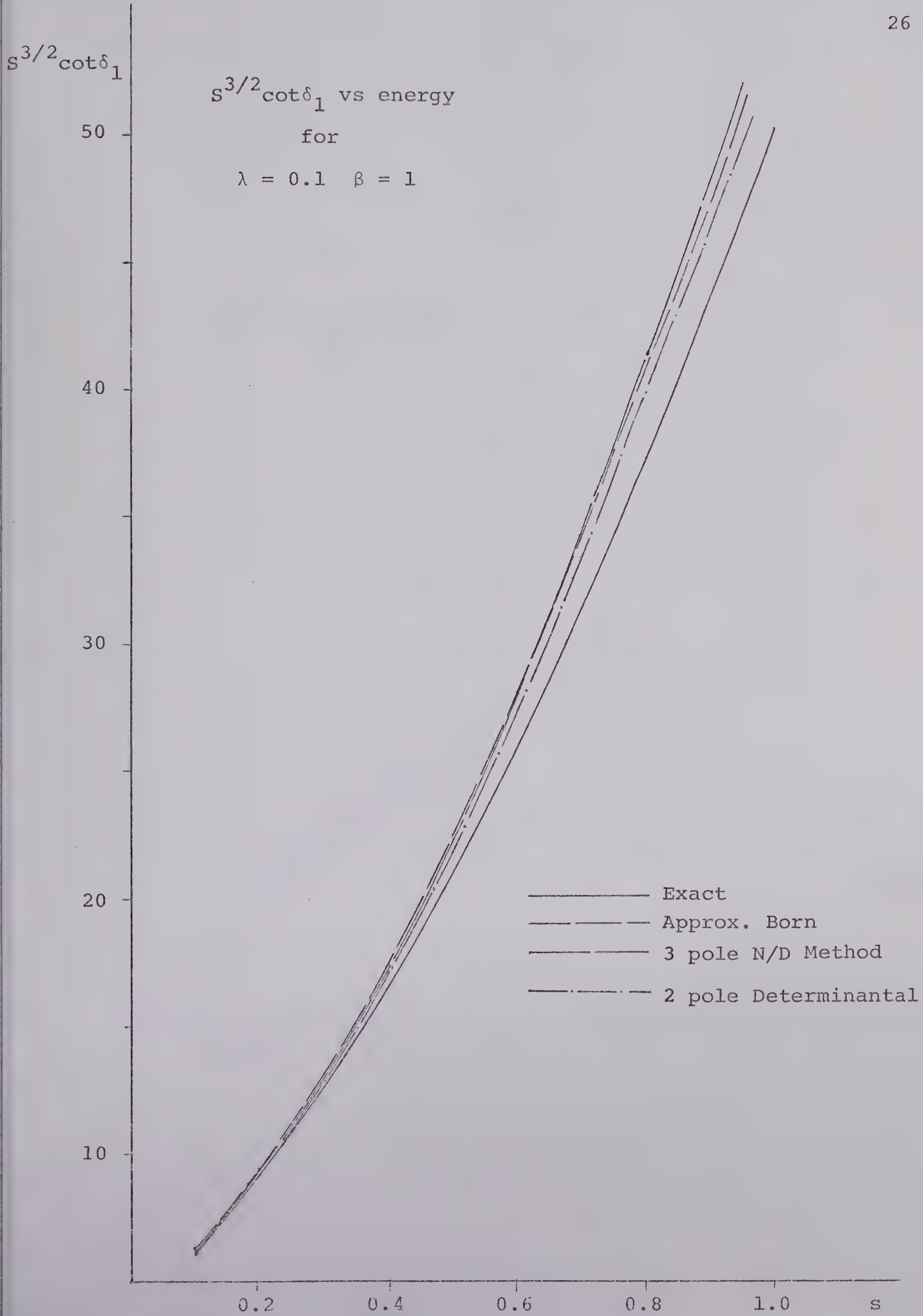
for $\lambda = 0.5$ $\beta = 1$

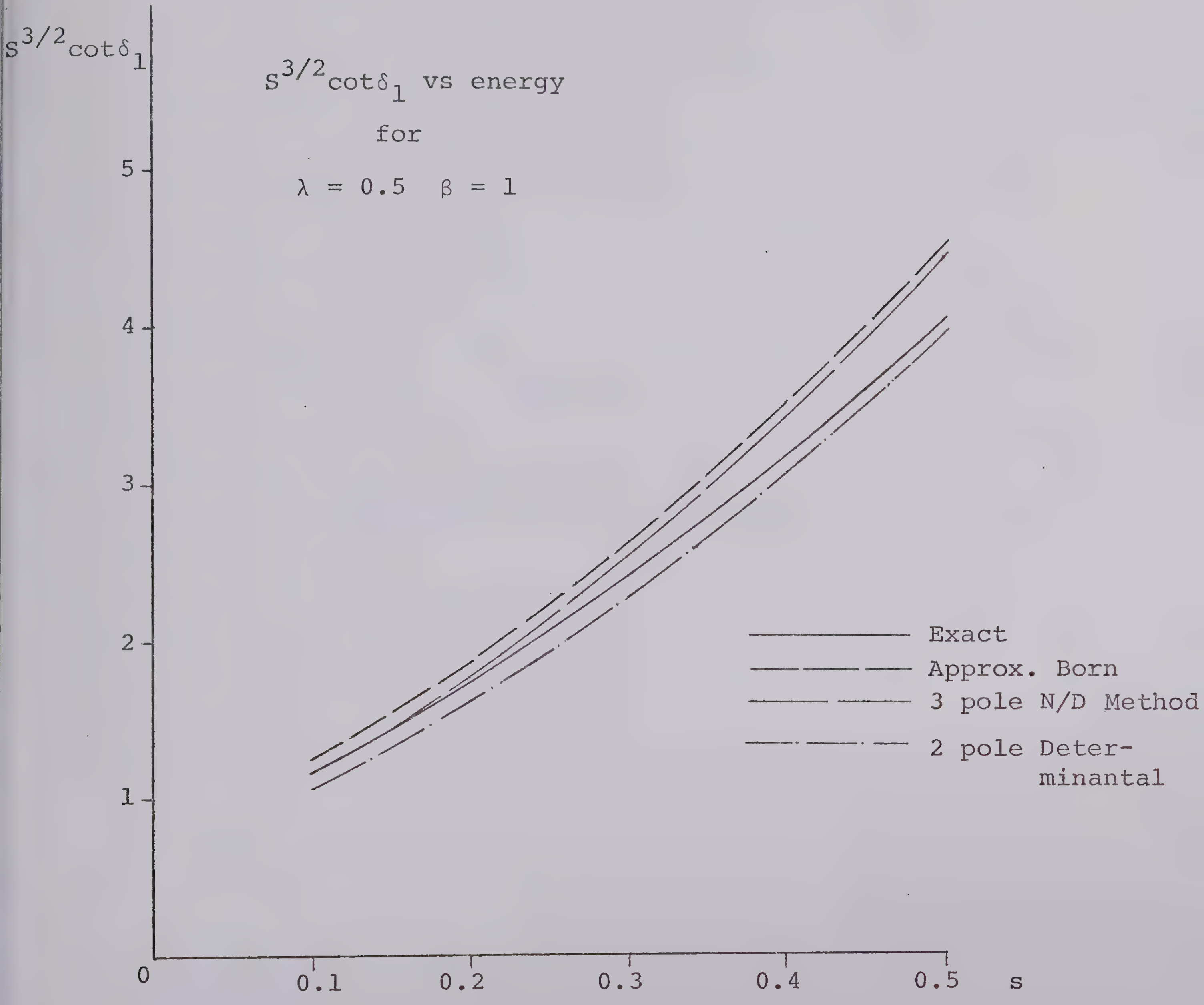
$s = 0.1$	1.26	1.06568	1.175	1.1784
$s = 0.2$	1.88	1.60366	1.783	1.7433
$s = 0.3$	2.63	2.26224	2.531	2.4128
$s = 0.4$	3.51	3.04082	3.414	3.1824
$s = 0.5$	4.5	3.9394	4.427	4.0502



$S^{3/2}\cot\delta_1$ vs energy
for
$\lambda = -0.1 \quad \beta = 1$


$S^{3/2}\cot\delta_1$
$S^{3/2}\cot\delta_1$ vs energy
for
$\lambda = -0.5 \quad \beta = 1$
-10
-8
-6
-4
-2
0.2
0.4
0.6
0.8
1.0
s
Exact
Approx. Born
3 pole N/D Method
2 pole Determinantal


$s^{3/2}\cot\delta_1$
$s^{3/2}\cot\delta_1$ vs energy
for
$\lambda = 0.1 \quad \beta = 1$
50
40
30
20
10
0.2
0.4
0.6
0.8
1.0
s
Exact
Approx. Born
3 pole N/D Method
2 pole Determinantal


$S^{3/2}\cot\delta_1$
$S^{3/2}\cot\delta_1$ vs energy
for
$\lambda = 0.5 \quad \beta = 1$
5
4
3
2
1
0
0.1
0.2
0.3
0.4
0.5
s
Exact
Approx. Born
3 pole N/D Method
2 pole Determinantal

PART II

APPLICATIONS OF VENEZIANO MODEL

Introduction

Three years ago, based on the assumption of the linealy rising Regge trajectories, G. Veneziano(1) proposed a remarkable formula which, despite its simple form, exhibits many important features of the strong interaction theory, namely, it

(i) satisfies the crossing symmetry,
(ii) gives the correct asymptotic behaviour,
(iii) satisfies duality,
(iv) predicts a family of parallel daughter trajectories.

Later, Lovelace(2), in his fascinating paper, applied the Veneziano model to the $\pi\pi$ scattering and 3π final state processes and obtained results coincident with the predictions of current algebra. This initial success led to the suggestion(3) of the close connection between Veneziano model and current algebra.

A further investigation by Kawarabayashi et al.(4) on the $\pi\pi$, πK and KK scatterings, revealed that good agreement between the s-wave scattering lengths obtained from the Veneziano model and that from current algebra existed in all cases except the $K\bar{K}$ case. The current algebra result in this case is suspect since the PCAC assumption is used in current algebra together with a linear extrapolation from the Adler zero point to the threshold for the process. Since the $K\bar{K}$ channel is not

exotic, one expects this extrapolation not to be smooth because of the presence of bound state contributions. The Veneziano model on the other hand, in the case of $K\bar{K}$ scattering, provides a highly non-linear extrapolation from the Adler zero point to the threshold. In the following chapter, this point is further elucidated in the evaluation of the s and p wave scattering lengths by an explicit partial wave projection of the Veneziano amplitudes for the $\pi\pi$, πK and $K\bar{K}$ (and KK) scatterings.

In Lovelace's paper, the Adler zero(5) was produced by demanding the gamma function in the denominator of the Veneziano amplitude to have a pole. From this constraint he obtained the intercept of the ρ trajectory at zero energy to be 0.485, while the experimental result is ≈ 0.57(6). Applying the same technique to the $\pi\pi \rightarrow \pi A_1$ reaction, Fayyazuddin and Riazuddin(7) derived the Weinberg mass formula(8) for m_{A_1}, m_ρ and m_π. Since there are two invariant amplitudes serving to define the complete amplitude for $\pi\pi \rightarrow \pi A_1$ process, an attempt is made in Chapter III to enforce the Adler zero in an alternative way namely by requiring that the numerators of the two invariant amplitudes cancel at the Adler zero point. The consequences that follow are discussed in that chapter.

The pion pole model treatment by Lovelace to the $\bar{p}n \rightarrow 3\pi$, $K \rightarrow 3\pi$ and $\eta \rightarrow 3\pi$ processes, was questioned by several

authors$^{(9)-(12)}$. As Sutherland$^{(9)}$ and Jacob et al.$^{(10)}$ had pointed out in their papers that the off-mass shell effect of the intermediate pion should be quite large; in the last chapter of this part, the $K_2^o \to \pi^+\pi^-\gamma$ and $K_2^o \to \gamma\gamma$ decays are considered in pole model in which the intermediate state is taken as an off-mass shell pion (or 0^- π like meson) rather than the on-mass shell pion.

CHAPTER II

VENEZIANO MODEL OF THE $\pi\pi$, πK AND $K\bar{K}$ SCATTERINGS

Section I : Introduction

The form of the scattering amplitude proposed by Veneziano(1) has a simple structure which demonstrates the Regge pole-resonance duality. It was not intended to use the amplitude at low energies. In a fascinating yet somewhat intriguing paper by Lovelace(2) the Veneziano amplitude for the pion-pion scattering was used in an unexpected direction, namely, at very low energies with some intriguing results. Among the many successful results obtained by Lovelace from the leading term of the Veneziano amplitude were, (i) that the imposition of the Adler zero demanded that the intercept of the ρ-trajectory at $t=0$ be ≈ 0.485 while there is experimental evidence from the charge-exchange data(6) that it is ≈ 0.57; (ii) the ratio of the s-wave scattering lengths a_o and a_2 was within 10% of the ratio predicted by Weinberg through current algebra(13). There were other successes of the Veneziano model for π-π scattering(3). Led by this initial success of Veneziano model to π-π scattering a great deal of work was done on other two-body scattering processes, for example, π-K and KK (and $K\bar{K}$) scattering with varying degree of success(4). The evidence most of these works were

seeking was the answer to the question; is there a connection between the Veneziano model and the predictions of current algebra. The conclusion one can arrive at after the dust has settled down is that the Veneziano model has been remarkably successful in reproducing the current algebra results (say, the scattering lengths) for process involving pions, like π-π scattering and, though not successful to the same extent, the π-K scattering. It has however provided an entirely different result for the $K\bar{K}$ scattering lengths compared to the current algebra results. An explanation for the success in $\pi\pi$ and πK case and the failure in $K\bar{K}$ case is that if one matches the Veneziano amplitude to the current algebra amplitude at the Adler zero point, the physical amplitude is obtained through rather a small extrapolation in the external mass (or equivalently s, t and u). Current algebra predictions use a linear extrapolation in s, t and u while for small s, t and u the gamma functions in Veneziano model also provide essentially a linear extrapolation. Thus one can understand why one has a remarkable success for $\pi\pi$ and πK scatterings. In the $K\bar{K}$ case the amount of extrapolation in s, t and u involved in moving out of the Adler zero point to their physical values is very large and the gamma functions provide a highly nonlinear extrapolation leading to extremely large scattering lengths.

Though the Veneziano amplitude has the Regge pole-resonance duality it is not obvious that the amplitude provides partial wave amplitudes with the correct threshold behaviour. In this chapter we have done an explicit partial wave projection of the Veneziano amplitudes for the $\pi\pi$, πK and KK (and $K\bar{K}$) scatterings. We have also evaluated the s and p wave scattering lengths for all these scattering processes.

Section II : $\pi\pi$ Scattering

In $\pi^+\pi^-\to\pi^+\pi^-$ scattering there are no u-channel resonances since this channel is exotic, having I=2. In terms of Regge trajectories this situation is brought about by the exchange degeneracy of the ρ and f trajectories. However, this reaction is symmetric under s and t exchange and one may write the Veneziano amplitude for $\pi^+\pi^-\to\pi^+\pi^-$ as$^{(2)(14)}$ (only the leading term has been retained),

$$A(s,t) = -\beta \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(1-\alpha(s)-\alpha(t))} . \tag{2.1}$$

With this form the right asymptotic behaviour $A(s,t)\xrightarrow[s\to\infty]{} s^{\alpha(t)}$ is also guaranteed. $\alpha(s)$ is the ρ-trajectory taken real and linear in s

$$\alpha(s) = \frac{1}{2} + \alpha'(s-m_\pi^2) \tag{2.2}$$

with

$$\alpha' = [2(m_\rho^2 - m_\pi^2)]^{-1} \quad . \tag{2.3}$$

β is identified as

$$\beta = 2\, g_{\rho\pi\pi}^2 \tag{2.4}$$

by going to the ρ-pole at $s = m_\rho^2$ and identifying the p wave contribution with the ρ pole Feynman diagram.

The three iso-spin amplitudes for the π-π scattering in the s channel are$^{(14)}$

$$M^0(s,t) = \frac{3}{2}\,[A(s,t) + A(s,u)] - \frac{1}{2}\,A(t,u) \tag{2.5}$$

$$M^1(s,t) = A(s,t) - A(s,u) \tag{2.6}$$

$$M^2(s,t) = A(t,u) \tag{2.7}$$

and s, t and u are the Mandelstam variables.

One can write A in an alternative form. Consider A(s,t) written out in terms of the Euler Beta function in its integral form$^{(15)}$.

$$\begin{aligned} A(s,t) &= -\,\beta(1-\alpha(s)-\alpha(t))B(1-\alpha(s),1-\alpha(t)) \\ &= -\,\beta(1-\alpha(s)-\alpha(t)) \int_0^1 x^{-\alpha(s)}(1-x)^{-\alpha(t)}dx \end{aligned} \tag{2.8}$$

$$\text{for} \quad \alpha(s),\alpha(t) < 1 \;,$$

where $B(1-\alpha(s),1-\alpha(t))$ is the Beta function

$$B(1-\alpha(s),1-\alpha(t)) = \frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} .$$

By taking a transformation $\xi = \frac{1}{x} - 1$, one can convert the finite range of integration to span the range $0 \to \infty$,

$$A(s,t) = -\beta(1-\alpha(s)-\alpha(t)) \int_0^\infty \frac{(1+\xi)^{\alpha(s)+\alpha(t)}}{(1+\xi)^2\,\xi^{\alpha(t)}}\, d\xi \quad . \tag{2.9}$$

Using the linearity of the ρ trajectory, one can write (2.9) as

$$A(s,t) = -\beta\alpha'(2m_\pi^2-s-t) \int_0^\infty \frac{(1+\xi)^{\alpha'(s+t-2m_\pi^2)}}{(1+\xi)\,\xi^{\frac{1}{2}+\alpha'(t-m_\pi^2)}}\, d\xi \quad . \tag{2.10}$$

Similar expressions for $A(s,u)$ and $A(t,u)$ can be obtained by appropriate substitution of t and u into (2.10).

The $\ell=0$ partial wave of $A(s,t)$ can be projected out as follows:

$$A_o(s,t) = \frac{1}{2} \int_{-1}^{1} A(s,t)\, d\cos\theta \tag{2.11}$$

with $\cos\theta = 1 + \frac{2t}{s-4m_\pi^2}$, (2.11) can be written as

$$A_o(s,t) = -\frac{1}{s-4m_\pi^2} \int_0^{-(s-4m_\pi^2)} A(s,t)dt$$

$$= \frac{\alpha'\beta}{s-4m_\pi^2} \int_0^{-(s-4m_\pi^2)} dt(2m_\pi^2-s-t)\left(\frac{1+\xi}{\xi}\right)^{\alpha' t} \times$$

$$\times \int_0^\infty d\xi \frac{(1+\xi)^{\alpha'(s-2m_\pi^2)}}{(1+\xi)\,\xi^{\frac{1}{2}-\alpha' m_\pi^2}} \quad . \tag{2.12}$$

We first do the t integration. The term with coefficient proportional to t^{o} is

$$\int_{0}^{-(s-4m_{\pi}^{2})} dt \left(\frac{1+\xi}{\xi}\right)^{\alpha' t} = \int_{0}^{-(s-4m_{\pi}^{2})} dt \exp[\alpha' t \ln \frac{1+\xi}{\xi}]$$

$$= \frac{1}{\alpha' \ln \frac{1+\xi}{\xi}} \{\exp[-\alpha'(s-4m_{\pi}^{2}) \ln \frac{1+\xi}{\xi}] - 1\} . \qquad (2.13)$$

For s close to the threshold, i.e., $s \to 4m_{\pi}^{2}$, (2.13) may be expanded into a series form as follows,

$$\int_{0}^{-(s-4m_{\pi}^{2})} dt \left(\frac{1+\xi}{\xi}\right)^{\alpha' t} = -(s-4m_{\pi}^{2})[1 - \frac{\alpha'}{2}(s-4m_{\pi}^{2}) \ln \frac{1+\xi}{\xi} + \ldots] . \qquad (2.14)$$

The term with coefficient t in eq. (2.12) is,

$$\int_{0}^{-(s-4m_{\pi}^{2})} dt \; t \left(\frac{1+\xi}{\xi}\right)^{\alpha' t} = [- \frac{s-4m_{\pi}^{2}}{\alpha' \ln \frac{1+\xi}{\xi}} - \left(\frac{1}{\alpha' \ln \frac{1+\xi}{\xi}}\right)^{2}] \times$$

$$\times \exp[-\alpha'(s-4m_{\pi}^{2}) \ln \frac{1+\xi}{\xi}] + \left(\frac{1}{\alpha' \ln \frac{1+\xi}{\xi}}\right)^{2} . \qquad (2.15)$$

In the limit $s \to 4m_{\pi}^{2}$, eq. (2.15) yields

$$(s-4m_{\pi}^{2})[\frac{1}{2} - \frac{1}{3} \alpha'(s-4m_{\pi}) \ln \frac{1+\xi}{\xi} + \ldots.] . \qquad (2.16)$$

Putting (2.14) and (2.16) into (2.11) one has

$$A_o(s,t) = \alpha'\beta \frac{s}{2} \int_0^\infty \frac{(1+\xi)^{\alpha'(s-2m_\pi^2)}}{(1+\xi)\,\xi^{\frac{1}{2}-\alpha' m_\pi^2}}\, d\xi$$

$$- \frac{\alpha'^2\beta}{6}(s+4m_\pi^2) \int_0^\infty \frac{(1+\xi)^{\alpha'(s-2m_\pi^2)}}{(1+\xi)\,\xi^{\frac{1}{2}-\alpha' m_\pi^2}} \ln\left(\frac{1+\xi}{\xi}\right) d\xi$$

or

$$A_o(s,t) = 2\alpha'\beta[k^2+m_\pi^2] \int_0^\infty \frac{(1+\xi)^{\alpha'(s-4m_\pi^2)+\alpha' 2m_\pi^2}}{(1+\xi)\,\xi^{\frac{1}{2}-\alpha' m_\pi^2}}\, d\xi$$

$$- \frac{4}{3}\alpha'^2\beta[2k^4+3m_\pi^2k^2] \int_0^\infty \frac{(1+\xi)^{\alpha'(s-4m_\pi^2)+\alpha' 2m_\pi^2}}{(1+\xi)\,\xi^{\frac{1}{2}-\alpha' m_\pi^2}} \times$$

$$\times [\ln(1+\xi) - \ln\xi]\, d\xi \quad , \qquad (2.17)$$

where $k^2 = \frac{1}{4}(s-4m_\pi^2)$ is the momentum of the pion in C.M. system. The factor in the integrand $(1+\xi)^{\alpha'(s-4m_\pi^2)}$ = $\exp[\alpha'(s-4m_\pi^2)\ln(1+\xi)]$ can be expanded in powers of $(s-4m_\pi^2)$ when $s \to 4m_\pi^2$. Keeping the terms of order α', one obtains the following

$$A_o(s,t) = 2\alpha'\beta[k^2+m_\pi^2] \int_0^\infty \frac{(1+\xi)^{2\alpha' m_\pi^2}}{(1+\xi)\,\xi^{\frac{1}{2}-\alpha' m_\pi^2}}\, d\xi$$

$$+ \frac{4\alpha'^2\beta}{3}[2k^4+3m_\pi^2k^2] \int_0^\infty \frac{(1+\xi)^{2\alpha' m_\pi^2}\ln\xi}{(1+\xi)\,\xi^{\frac{1}{2}-\alpha' m_\pi^2}}\, d\xi$$

$$+ \frac{4}{3}\alpha'^2\beta[4k^4+3m_\pi^2k^2] \int_0^\infty \frac{(1+\xi)^{2\alpha' m_\pi^2}\ln(1+\xi)}{(1+\xi)\,\xi^{\frac{1}{2}-\alpha' m_\pi^2}}\, d\xi . \qquad (2.18)$$

To simplify the above integrations, one can make the following approximation. For $\alpha' m^2 \approx \frac{1}{60} << \frac{1}{2}$, one may neglect the factor $(1+\xi)^{2\alpha' m_\pi^2/\xi \; -\alpha' m_\pi^2}$ in the integrand, then the above integrals give$^{(15)}$

$$\left.\begin{array}{l} \displaystyle\int_0^\infty \frac{d\xi}{(1+\xi)\xi^{\frac{1}{2}}} = \pi \\[2em] \displaystyle\int_0^\infty \frac{d\xi \;\; \ln \xi}{(1+\xi)\xi^{\frac{1}{2}}} = 0 \\[2em] \displaystyle\int_0^\infty \frac{d\xi \;\; \ln(1+\xi)}{(1+\xi)\xi^{\frac{1}{2}}} = 2\pi \; \ln 2 \quad , \end{array}\right\} \qquad (2.19)$$

and $A_o(s,t)$ becomes

$$A_o(s,t) = \pi\alpha'\beta[2k^2 + 2m_\pi^2 + \tfrac{8}{3}\,\alpha'(4k^4 + 3k^2m_\pi^2)\;\ln 2] \;. \qquad (2.20)$$

The $\ell=0$ partial wave of $A(s,u)$ is given by

$$A_o(s,u) = \frac{1}{2}\int_{-1}^{1} A(s,u)\; d\,\cos\theta$$

$$= -\frac{1}{2}\int_0^{-(s-4m_\pi^2)} du\; A(s,u)\,\left(\frac{2}{s-4m_\pi^2}\right)$$

$$= A_o(s,t) \quad . \qquad (2.21)$$

Therefore,

$$A_o(s,u) = \pi\alpha'\beta[2k^2 + 2m_\pi^2 + \tfrac{8}{3}\,\alpha'(4k^4 + 3k^2m_\pi^2)\;\ln 2] \,. \qquad (2.22)$$

The $\ell=0$ partial wave, $A_o(u,t)$, of $A(u,t)$ is projected out in a similar way as follows:

$$A_o(u,t) = \frac{1}{2}\int_{-1}^{1} A(u,t)\ d\cos\theta$$

$$= -\frac{1}{2}\int_{0}^{-(s-4m_\pi^2)} dt\ A(u,t)\left(\frac{2}{s-4m_\pi^2}\right)$$

$$= -\pi\alpha'\beta[4k^2+ 2m_\pi^2- 16\alpha'(2k^4+ k^2m_\pi^2)\ln 2]\ . \qquad (2.23)$$

Now using eqs. (2.5)-(2.7), the s wave isospin amplitudes for $\pi\pi$ scattering can be written down,

$$M_o^o(s,t) = \frac{3}{2}\ [A_o(s,t) + A_o(s,u)] - \frac{1}{2}\ A_o(t,u)$$

$$= \pi\alpha'\beta[8k^2+7m_\pi^2+16\alpha'(k^4+k^2m_\pi^2)\ln 2 + \ldots] \qquad (2.24)$$

$$M_o^1(s,t) = A_o(s,t) - A_o(s,u) = 0 \qquad (2.25)$$

$$M_o^2(s,t) = A_o(u,t) = -2\pi\alpha'\beta[2k^2+m_\pi^2-8\alpha'(2k^4+k^2m_\pi^2)\ln 2 + \ldots]. \qquad (2.26)$$

The remarkable feature of equations (2.24) and (2.26) is that we get the ratio of s wave scattering lengths $a_o/a_2 = -7/2$ which is precisely the current algebra value$^{(13)}$. A more remarkable feature yet is that to order α' even the effective range term (terms proportional to k^2) are precisely those given by current algebra$^{(16)}$.

The p wave projections of A(s,t) and A(s,u) can be calculated in the same manner,

$$A_1(s,t) = \frac{1}{2} \int_{-1}^{1} A(s,t) \cos\theta \, d \cos\theta$$

$$= - \frac{1}{s-4m_\pi^2} \int_0^{-(s-4m_\pi^2)} dt \, A(s,t) (1 + \frac{2t}{s-4m_\pi^2})$$

$$= \frac{\alpha'\beta}{s-4m_\pi^2} \int_0^{-(s-4m_\pi^2)} dt (2m_\pi^2 - s - t) (1 + \frac{2t}{s-4m_\pi^2}) \int_0^\infty d\xi \times$$

$$\times \frac{(1+\xi)^{\alpha'(s+t-2m_\pi^2)}}{(1+\xi)\,\xi^{\frac{1}{2}-\alpha' t-\alpha' m_\pi^2}} \; . \qquad (2.27)$$

In (2.27), the term of order t^2 can be evaluated as follows:

$$\int_0^{-(s-4m_\pi^2)} t^2 \exp[\alpha' t \ln \frac{1+\xi}{\xi}] \, dt$$

$$= \{\exp[-(s-4m_\pi^2) \ln \frac{1+\xi}{\xi}]\} [\frac{(s-4m_\pi^2)^2}{\ln \frac{1+\xi}{\xi}} + \frac{2(s-4m_\pi^2)}{(\ln \frac{1+\xi}{\xi})^2} + \frac{2}{(\ln \frac{1+\xi}{\xi})^3}]$$

$$- \frac{2}{(\ln \frac{1+\xi}{\xi})^3} \; . \qquad (2.28)$$

In the limit $s \to 4m_\pi^2$, one expands $\exp[-(s-4m_\pi^2) \ln \frac{1+\xi}{\xi}]$ in a power series in $(s-4m_\pi^2)$ and obtains

$$\int_0^{-(s-4m_\pi^2)} t^2 dt \exp[\alpha' t \ln \frac{1+\xi}{\xi}]$$

$$= -\frac{1}{3}(s-4m_\pi^2)^3[1 - \frac{3}{4}(s-4m_\pi^2) \ln \frac{1+\xi}{\xi} + \ldots] . \qquad (2.29)$$

After all the integrations in (2.27) have been carried out, one gets

$$A_1(s,t) = \frac{\pi}{3}\alpha'\beta[2k^2+8\alpha'(2k^4+k^2m_\pi^2) \ln 2] \qquad (2.30)$$

$A_1(s,u)$ can be shown to be equal to $-A_1(s,t)$

$$A_1(s,u) = \frac{1}{2}\int_{-1}^{1} A(s,u) \cos\theta d \cos\theta$$

$$= \frac{1}{s-4m_\pi^2}\int_0^{-(s-4m_\pi^2)} du\, A(s,u)(1 + \frac{2u}{s-4m_\pi^2})$$

$$= -A_1(s,t) \qquad (2.31)$$

and

$$A_1(u,t) = 0 . \qquad (2.32)$$

Therefore, for the p-wave amplitude, only the I=1 isospin amplitude is non-zero and has a value,

$$M_1^1(s,t) = A_1(s,t) - A_1(s,u)$$

$$= \frac{4}{3}\pi\alpha'\beta[k^2+4\alpha'(2k^4+k^2m_\pi^2)\ln 2 + \ldots] . \qquad (2.33)$$

The scattering lengths can be determined from the amplitudes (2.24), (2.26) and (2.33).

(i) $$32\pi a_1^0 m_\pi = [M_0^0(s,t)]_{k=0}$$

$$a_0^0 \approx 0.175\ m_\pi^{-1} \tag{2.34}$$

(ii) $$32\pi a_0^2 m_\pi = [M_0^2(s,t)]_{k=0}$$

$$a_0^2 \approx -0.05\ m_\pi^{-1} \tag{2.35}$$

(iii) $$32\pi a_1^1 m_\pi = \left[\frac{M_1^1(s,t)}{k^2}\right]_{k=0}$$

$$a_1^1 = 0.033\ m_\pi^{-3}\ . \tag{2.36}$$

All these results are in excellent agreement with current algebra results(13). The conclusion one can draw from these calculations is that the Veneziano amplitude for π-π scattering (with the leading term) has the correct threshold behaviour in the low partial waves and that the amplitude has a remarkable likeness to the current algebra amplitude at low energies.

Section III : $\pi K \rightarrow \pi K$ Scattering

Consider the elastic scattering

$$\pi(p) + K(q) \text{ —— } \pi(p') + K(q')\ ,$$

where p, q, p' and q' are the four momenta of the particles. The Mandelstam variables s, t and u in the centre of mass system are

$$s = (p+q)^2 = (p_o+q_o)^2 \tag{2.37}$$

$$t = (p-p')^2 = -2\underset{\sim}{p}^2(1-\cos\theta) \tag{2.38}$$

$$u = 2m_\pi^2+2M_K^2-s+2\underset{\sim}{p}^2(1-\cos\theta) \ . \tag{2.39}$$

The isospin amplitudes in the t channel ($\pi\pi\to K\bar{K}$ reaction) are$^{(14)}$

$$T^o(s,t,u) = \lambda'[V_{K^*\rho}(s,t) + V_{K^*\rho}(u,t)] \tag{2.40}$$

$$T^1(s,t,u) = \beta'[V_{K^*\rho}(s,t) - V_{K^*\rho}(u,t)] \ . \tag{2.41}$$

The Veneziano amplitude with the correct asymptotics is given by$^{(14)(17)}$

$$V_{K^*\rho}(s,t) = \frac{\Gamma(1-\alpha_\rho(t))\Gamma(1-\alpha_{K^*}(s))}{\Gamma(1-\alpha_\rho(t) - \alpha_{K^*}(s))} \ , \tag{2.42}$$

where α_ρ and α_{K^*} are the ρ and K^* trajectories. Assuming universal slope and imposing the Adler zeros we get

$$\alpha_\rho(t) = \frac{1}{2} + \alpha'(t-m_\pi^2) \tag{2.43}$$

$$\alpha_{K^*}(s) = \frac{1}{2} + \alpha'(s-m_K^2) \ . \tag{2.44}$$

The slope $\alpha' = \frac{1}{2}(m_\rho^2 - m_\pi^2)^{-1}$. (2.45)

The s channel isospin amplitudes are obtained from (2.40) and (2.41) through the crossing matrix[(14)].

$$M_{st} = \frac{1}{2}\begin{pmatrix} \sqrt{\frac{2}{3}} & 2 \\ \sqrt{\frac{2}{3}} & -1 \end{pmatrix} .$$

By imposing the condition $\lambda' = \sqrt{\frac{3}{2}}\,\beta'$, the exotic $I=3/2$ s channel pole is removed. The s channel isosopin 1/2 and 3/2 amplitudes are then

$$S^{1/2}(s,t,u) = \frac{\beta'}{2}\,(3V_{\rho K*}(t,s)-V_{\rho K*}(t,u)) \qquad (2.46)$$

$$S^{3/2}(s,t,u) = \beta' V_{\rho K*}(t,u) \qquad (2.47)$$

with

$$\beta' = -2g_{\rho\pi\pi}g_{\rho K\bar{K}} = -g_{\rho\pi\pi}^2 \; . \qquad (2.48)$$

As in the previous section the Veneziano amplitude can be written as

$$V_{K*\rho}(s,t) = [1-\alpha_\rho(t)-\alpha_{K*}(s)]\int_0^\infty \frac{(1+\xi)^{\alpha_{K*}(s)+\alpha_\rho(t)}}{(1+\xi)^2\,\xi^{\alpha_\rho(t)}}\,d\xi$$

or

$$V_{K*\rho}(s,t) = \alpha'[m_K^2+m_\pi^2-s-t]\int_0^\infty \frac{(1+\xi)^{\alpha'(s+t-m_K^2-m_\pi^2)}}{(1+\xi)\,\xi^{\frac{1}{2}+\alpha'(t-m_\pi^2)}}\,d\xi . \qquad (2.49)$$

The s wave amplitude $V_{K^*\rho}(s,t)_{\ell=0}$ is then projected out via

$$V_{K^*\rho}(s,t)_{\ell=0} = \frac{1}{2}\int_{-1}^{1} V_{K^*\rho}(s,t)d\cos\theta. \tag{2.50}$$

Since $\cos\theta = 1 + \frac{t}{2p^2}$ (2.51)

and

$$p^2 = \frac{s^2-2s(m_\pi^2+m_K^2) + (m_\pi^2-m_K^2)^2}{2s} \tag{2.52}$$

one has

$$V_{K^*\rho}(s,t)_{\ell=0} = -\frac{1}{4p^2}\int_0^{-4p^2} V_{K^*\rho}(s,t)dt$$

$$= -\frac{\alpha'}{4p^2}\int_0^{-4p^2} dt(m_K^2+m_\pi^2-s-t)\exp[\alpha' t \ln\frac{1+\xi}{\xi}]\int_0^\infty d\xi \times$$

$$\times \frac{(1+\xi)^{\alpha'(s-m_K^2-m_\pi^2)}}{(1+\xi)\xi^{\frac{1}{2}-\alpha' m_\pi^2}} .$$

After the integrations have been worked out, in the region where $s \to (m_K+m_\pi)^2$ and $4p^2 \to 0$, the result one gets is

$$V_{K^*\rho}(s,t)_{\ell=0} = -\alpha'\int_0^\infty \frac{(1+\xi)^{2\alpha' m_K m_\pi}}{(1+\xi)\xi^{\frac{1}{2}-\alpha' m_\pi^2}} d\xi\{[(s-m_K^2-m_\pi^2)-2p^2]$$

$$- [(s-m_K^2-m_\pi^2)2p^2 - \frac{1}{3}(4p^2)^2 - (s-m_K^2-m_\pi^2)(s-(m_K+m_\pi)^2)$$

$$+ 2p^2(s-(m_K+m_\pi)^2)]\alpha' \ln(1+\xi)$$

$$+ [(s-m_K^2-m_\pi^2)2p^2 - \frac{1}{3}(4p^2)^2]\alpha' \ln\xi\} . \tag{2.53}$$

As $\alpha' m_\pi \approx \frac{1}{60}$ and $2\alpha' m_K m_\pi \approx \frac{1}{8}$, we can neglect these exponents in the integrand. The integration can then be reduced to the same simple form as in (2.19). After the ξ integration has been done, we end up with the expression

$$V_{K^*\rho}(s,t)_{\ell=0} = -\alpha'\pi \left[\frac{s^2-(m_K^2-m_\pi^2)^2}{2s}\right]$$

$$- \alpha'^2\pi\frac{\ln 2}{3s}\{2s^4+s^3[(m_K^2-m_\pi^2)-3(m_K+m_\pi)^2]$$

$$+ s^2 \; [\; 8m_K^2 m_\pi^2 \;] - s[5(m_K^2+m_\pi^2)(m_K^2-m_\pi^2)^2$$

$$- 3(m_K^2-m_\pi^2)^2(m_K+m_\pi)^2] + 2(m_K^2-m_\pi^2)^4\} \; . \qquad (2.54)$$

Taking the leading term, i.e., term of order α' in (2.54), we have

$$V_{K^*\rho}(s,t)_{\ell=0} \approx -2\alpha' \; m_K m_\pi \qquad (2.55)$$

at threshold.

Following the same procedure, the s-wave projection of $V_{K^*\rho}(t,u)$ is

$$V_{K^*\rho}(t,u)_{\ell=0} = \alpha'\pi(s-m_K^2-m_\pi^2)[1 - 2\alpha'(s-(m_K+m_\pi)^2)\ln 2] \qquad (2.56)$$

or $\quad V_{K^*\rho}(t,u)_{\ell=0} \approx 2\alpha'\pi m_K m_\pi \quad$ at threshold.

Then from (2.46) and (2.47), the s channel isospin amplitudes at threshold are,

$$S_o^{1/2}(s,t,u) = \frac{\beta'}{2}[3V_{K^*\rho}(s,t)_{\ell=0} - V_{K^*\rho}(t,u)_{\ell=0}]$$

$$= -4\alpha'\beta'\pi m_K m_\pi \tag{2.57}$$

$$S_o^{3/2}(s,t,u) = \beta' V_{K^*\rho}(t,u)_{\ell=0} = 2\alpha'\beta'\pi m_K m_\pi \ . \tag{2.58}$$

The scattering lengths for πK scattering are given by the following relations

$$8\pi a_o^{1/2}\sqrt{s} = S_o^{1/2}(s,t,u)\Big|_{\sqrt{s}\approx m_K}$$

$$\therefore\ a_o^{1/2} \approx 0.2\ m_\pi^{-1} \tag{2.59}$$

and

$$8\pi a_o^{3/2}\sqrt{s} = S_o^{3/2}(s,t,u)\Big|_{\sqrt{s}\approx m_K}$$

$$\therefore\ a_o^{3/2} \approx -0.10\ m_\pi^{-1} . \tag{2.60}$$

Both results are in good agreement with the current algebra predictions(18).

For the p-wave projections of $V_{K^*\rho}(s,t)$ and $V_{K^*\rho}(u,t)$ we have the following

$$V_{K^*\rho}(s,t)_{\ell=1} = -\alpha'\pi\ \frac{4p^2}{3}[\frac{1}{2} + \alpha' 2(s-m_K^2-m_\pi^2-m_K m_\pi-2p^2)\ln 2] \tag{2.61}$$

and

$$V_{K^*\rho}(u,t)_{\ell=1} = 0 \ . \tag{2.62}$$

The p wave isospin amplitudes are given by

$$S_1^{1/2}(s,t,u) = \alpha'\beta'\pi p^2 + \alpha'^2\beta'\pi \ln 2[4p^2(s-m_K^2-m_\pi^2-m_K m_\pi-2p^2)] \tag{2.63}$$

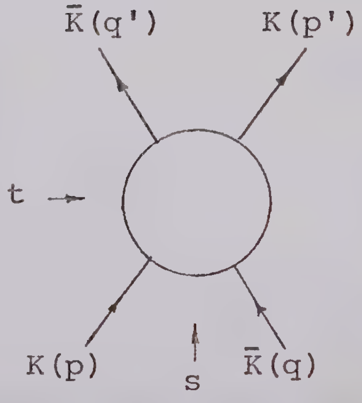
$$S_1^{3/2}(s,t,u) = 0 \ . \tag{2.64}$$

The scattering lengths obtained from (2.63) and (2.64) are then

$$a_1^{1/2} \approx 0.0125\ m_\pi^{-3} \tag{2.65}$$

$$a_1^{3/2} \approx 0 \ . \tag{2.66}$$

Section IV : KK and KK̄ Scattering



Consider the KK̄ scattering

$$K(p) + \bar{K}(q) \longrightarrow K(p') + \bar{K}(q') \ ,$$

where p, q, p' and q' are the particles' four momenta. Since there are no resonances in the t channel (being exotic), we write the t channel amplitude as(14)

$$T^0(s,t,u) = \alpha''(V_{\rho\phi}(u,s) - V_{\rho\phi}(s,u)) \tag{2.67}$$

$$T^1(s,t,u) = \beta''(V_{\rho\phi}(u,s) + V_{\rho\phi}(s,u)) \ . \tag{2.68}$$

Here the Veneziano amplitude is

$$V_{\rho\phi}(u,s) = \frac{\Gamma(1-\alpha_\rho(u))\Gamma(1-\alpha_\phi(s))}{\Gamma(1-\alpha_\rho(u) - \alpha_\phi(s))} \tag{2.69}$$

and $\alpha_\rho(u)$ and $\alpha_\phi(s)$ are the ρ and ϕ trajectories with the universal slope $\alpha' = \alpha'_\rho$.

$$\alpha_\rho(u) = \frac{1}{2} + \alpha'(u-m_\pi^2) \tag{2.70}$$

$$\alpha_\phi(s) = \frac{1}{2} + \alpha'(s-2m_K^2+m_\pi^2) \ . \tag{2.71}$$

In the centre of mass system the Mandelstam variables s, t and u are defined as

$$\begin{cases} s = (p+q)^2 = (p_o+q_o)^2 & (2.72) \\ t = (p-q')^2 = 4m_K^2-s+2\underset{\sim}{p}^2(1-\cos\theta) & (2.73) \\ u = (p-p')^2 = -2\underset{\sim}{p}^2(1-\cos\theta) & (2.74) \\ s + t + u = 4m_K^2 & (2.75) \end{cases}$$

where the three momentum of the K meson $\underset{\sim}{p}$ is given by

$$\underset{\sim}{p}^2 = \frac{s - 4m_K^2}{4} \ . \tag{2.76}$$

The s channel amplitudes are determined from the crossing matrix$^{(14)}$

$$M_{st} = \frac{1}{2}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} .$$

To exclude the ϕ pole in the S^1 amplitude, we impose the condition that $\beta'' = -\alpha'$, hence

$$S^O = \beta''(2V_{\rho\phi}(u,s) + V_{\rho\phi}(s,u)) \tag{2.77}$$

$$S^1 = \beta'' V_{\rho\phi}(s,u) \tag{2.78}$$

and β'' can be identified as $= -4g^2_{\rho K\bar{K}}$.

In the case of $K\bar{K}$ scattering we cannot write

$$V_{\rho\phi}(s,u) = (1-\alpha_\rho(s)-\alpha_\phi(u))B(1-\alpha_\rho(s),\ 1-\alpha_\phi(u))$$

with

$$B(1-\alpha_\rho(s),\ 1-\alpha_\phi(u)) = \int_0^1 x^{-\alpha_\rho(s)} (1-x)^{-\alpha_\phi(u)}\ dx ,$$

because this integral form is valid only for $1-\alpha_\rho(s) > 0$ and $1-\alpha_\phi(u) > 0$. In the present case we have $1-\alpha_\rho(s) = \frac{1}{2} - \alpha'(s-m_\pi^2) < 0$ even at threshold $s = 4m_K^2$. We may however modify $V_{\rho\phi}(s,u)$ in the following way

$$\begin{aligned} V_{\rho\phi}(s,u) &= (1-\alpha_\rho(s)-\alpha_\phi(u)) \frac{\Gamma(1-\alpha_\rho(s)\Gamma(1-\alpha_\phi(u))}{\Gamma(2-\alpha_\rho(s) - \alpha_\phi(u))} \\ &= (1-\alpha_\rho(s)-\alpha_\phi(u)) \frac{2-\alpha_\rho(s)-\alpha_\phi(u)}{1-\alpha_\rho(s)} \frac{\Gamma(2-\alpha_\rho(s))\Gamma(1-\alpha_\phi(u))}{\Gamma(3-\alpha_\rho(s) - \alpha_\phi(u))} \\ &= (1-\alpha_\rho(s)-\alpha_\phi(u)) \frac{2-\alpha_\rho(s)-\alpha_\phi(u)}{1-\alpha_\rho(s)} B(2-\alpha_\rho(s),1-\alpha_\phi(u)) \end{aligned} \tag{2.79}$$

Now, we can express B in the form

$$B(2-\alpha_\rho(s),1-\alpha_\phi(u)) = \int_0^1 x^{1-\alpha_\rho(s)} (1-x)^{-\alpha_\phi(u)} dx . \qquad (2.80)$$

The requirements $1-\alpha_\phi(u) > 0$ and $2-\alpha_\rho(s) > 0$ are now satisfied. Using the transformation $\xi = \frac{1}{x} - 1$, the expression (2.79) is changed into the form,

$$\begin{aligned} V_{\rho\phi}(s,u) &= [1 - \frac{1-\alpha_\phi(u)}{1-\alpha_\rho(s)}] (2-\alpha_\rho(s)-\alpha_\phi(u)) \int_0^\infty \frac{(1+\xi)^{\alpha_\rho(s)+\alpha_\phi(u)-1}}{(1+\xi)^2 \xi^{\alpha_\phi(u)}} d\xi \\ &= [\frac{\alpha'(s+u-2m_K^2)}{\frac{1}{2} - \alpha'(s-m_\pi^2)}][1 + \alpha'(2m_K^2-s-u)] \int_0^\infty \times \\ &\quad \times \frac{(1+\xi)^{\alpha'(s+u-2m_K^2)} d\xi}{(1+\xi)^2 \xi^{\frac{1}{2}+\alpha'(u-2m_K^2+m_\pi^2)}} . \qquad (2.81) \end{aligned}$$

With the integral form of $V_{\rho\phi}(s,u)$ determined in (2.81) we are ready to project out the s and p wave components of $V_{\rho\phi}(s,u)$.

The s wave amplitude $V_{\rho\phi}(s,u)_{\ell=0}$ is given by

$$V_{\rho\phi}(s,u)_{\ell=0} = \frac{1}{2} \int_{-4p^2}^{0} V_{\rho\phi}(s,u) \frac{du}{2p^2} .$$

Treating the u integration as in the previous sections we have up to the order α'^2 the expression

$$\begin{aligned} V_{\rho\phi}(s,u)_{\ell=0} &= \frac{\alpha'}{\frac{1}{2}-\alpha'(s-m_\pi^2)} \Big\{ \int_0^\infty \Big[-\frac{s}{2} + \frac{\alpha'}{3}(s^2-2m_K^2 s+4m_K^4) \Big] \times \\ &\times \frac{(1+\xi)^{2\alpha' m_K^2}\, d\xi}{(1+\xi)^2 \xi^{\frac{1}{2}-\alpha'(2m_K^2-m_\pi^2)}} - \int_0^\infty \frac{\alpha'}{3}(s-4m_K^2)(s-m_K^2) \times \\ &\times \frac{(1+\xi)^{2\alpha' m_K^2} \ln(1+\xi)\, d\xi}{(1+\xi)^2 \xi^{\frac{1}{2}} \xi^{-(2\alpha' m_K^2-\alpha' m_\pi^2)}} - \int_0^\infty \frac{\alpha'}{6}(s-4m_K^2)(s+2m_K^2) \times \\ &\times \frac{(1+\xi)^{2\alpha' m_K^2} \ln \xi \ d\xi}{(1+\xi)^2 \ \xi^{\frac{1}{2}-\alpha'(2m_K^2-m_\pi^2)}} \Big\} \ . \end{aligned} \tag{2.82}$$

The value $2\alpha' m_K^2 \approx \frac{1}{2}$ is quite large and cannot be ignored as we have done in the case for $2\alpha' m_\pi^2$. The various integrations in (2.82) can be evaluated approximately as follows:

$$\begin{aligned} \int_0^\infty d\xi \ \frac{(1+\xi)^{2\alpha' m_K^2}}{(1+\xi)^2 \ \xi^{\frac{1}{2}-\alpha'(2m_K^2-m_\pi^2)}} &\approx \int_0^\infty d\xi \ \frac{1}{(1+\xi)^{3/2}} = 2 \\ \int_0^\infty d\xi \ \frac{(1+\xi)^{2\alpha' m_K^2} \ln\ (1+\xi)}{(1+\xi)^2 \ \xi^{\frac{1}{2}-\alpha'(2m_K^2-m_\pi^2)}} &\approx \int_0^\infty d\xi \ \frac{\ln\ (1+\xi)}{(1+\xi)^{3/2}} \ . \end{aligned} \tag{2.83}$$

With the transformation $\xi = \frac{1}{x^2} - 1$, the last integral can be shown to be

$$\int_0^1 dx \ (2 \ln \frac{1}{x^2}) = 4 \ . \tag{2.84}$$

$$\int_0^\infty d\xi \, \frac{(1+\xi)^{2\alpha' m_K^2} \ln \xi}{(1+\xi)^2 \xi^{\frac{1}{2}-\alpha'(2m_K^2-m_\pi^2)}} \approx \int_0^\infty d\xi \, \frac{\ln \xi}{(1+\xi)^{3/2}}$$

and with the transformation $\xi = \frac{1}{x^2} - 1$, this is

$$2 \int_0^1 \ln(1-x^2)dx + 2 \int_0^1 dx \ln \frac{1}{x^2} = 4 \ln 2 \, . \qquad (2.85)$$

With eqs. (2.83)-(2.85), the final form of $V_{\rho\phi}(s,u)_{\ell=0}$ is obtained as

$$V_{\rho\phi}(s,u)_{\ell=0} = \frac{\alpha'}{\frac{1}{2}-\alpha'(s-m_\pi^2)} \{[-s+\frac{2}{3}\alpha'(s^2-2m_K^2 s+4m_K^4)$$

$$-\frac{4}{3}\alpha'(s-4m_K^2)(s-m_K^2) - \frac{2\alpha'}{3}(s-4m_K^2)(s+2m_K^2)\ln 2]\}. \qquad (2.86)$$

The p-wave amplitude $V_{\rho\phi}(s,u)_{\ell=1}$ is

$$V_{\rho\phi}(s,u)_{\ell=1} = \frac{1}{2}\int_{4p^2}^{0} V_{\rho\phi}(s,u)(1+\frac{u}{2p^2})\frac{1}{2p^2}du \qquad (2.87)$$

where $V_{\rho\phi}(s,u)$ is given by (2.81). The calculation is quite similar to the s wave calculation and the result is

$$V_{\rho\phi}(s,u)_{\ell=1} = \frac{\alpha'}{\frac{1}{2}-\alpha'(s-m_\pi^2)}(s-4m_K^2) \times$$

$$\times [-\frac{1}{3}+\frac{\alpha' s}{3}-\frac{2}{3}\alpha'(s-2m_K^2)-\frac{4\alpha'}{3}m_K^2 \ln 2]. \qquad (2.88)$$

To calculate the s and p wave components of the amplitude $V_{\rho\phi}(u,s)$, we first write down the integral form for $V_{\rho\phi}(u,s)$

$$V_{\rho\phi}(u,s) = (1-\alpha_\rho(u)-\alpha_\phi(s)) \frac{2-\alpha_\rho(u)-\alpha_\phi(s)}{1-\alpha_\phi(s)} \int_0^\infty \times$$

$$\times \frac{(1+\xi)^{\alpha_\phi(s)+\alpha_\rho(u)-1}}{(1+\xi)^2 \, \xi^{\alpha_\rho(u)}} \, d\xi$$

$$= \alpha'(2m_K^2-s-u)\left[\frac{1+\alpha'(2m_K^2-s-u)}{\frac{1}{2}-\alpha'(s-2m_K^2+m_\pi^2)}\right] \int_0^\infty \times$$

$$\times \frac{(1+\xi)^{\alpha'(s+u-2m_K^2)}}{(1+\xi)^2 \, \xi^{\frac{1}{2}+\alpha'(u-m_\pi^2)}} \, d\xi \quad , \tag{2.89}$$

then

$$V_{\rho\phi}(u,s)_{\ell=0} = \frac{1}{2} \int_{-4p^2}^{0} V_{\rho\phi}(u,s) \frac{du}{2p^2}$$

$$= \frac{\alpha'}{[\frac{1}{2}-\alpha'(s-2m_K^2+m_\pi^2)]} \{[-s + \frac{2}{3}\alpha'(s^2-2m_K^2 s+4m_K^4)$$

$$- \frac{4}{3}\alpha'(s-4m_K^2)(s-m_K^2) + 2\alpha' s(s-4m_K^2)\ln 2]\} \; . \tag{2.90}$$

$$V_{\rho\phi}(u,s)_{\ell=1} = -\frac{1}{4p^2}\int_0^{-4p^2} V_{\rho\phi}(u,s)\left(1+\frac{u}{2p^2}\right)du$$

$$= \frac{\alpha'(s-4m_K^2)}{\frac{1}{2}-\alpha'(s-2m_K^2+m_\pi^2)}\left\{-\frac{1}{3}-\frac{\alpha'}{3}[s-4m_K^2-2(s-4m_K^2)\ln 2]\right\}. \qquad (2.91)$$

Using eqs. (2.77) and (2.88), we can calculate the s and p wave isospin amplitudes for the $K\bar{K}$ scattering.

$$S_0^0(K\bar{K}) = \beta''\alpha'[-s-\frac{2}{3}\alpha'(s^2-8m_K^2s+4m_K^2)] \times$$

$$\times \left[\frac{1}{\frac{1}{2}-\alpha'(s-m_\pi^2)}+\frac{2}{\frac{1}{2}-\alpha'(s-2m_K^2+m_\pi^2)}\right]$$

$$+\frac{2}{3}\alpha'^2\beta''\ln 2\left[-\frac{(s+2m_K^2)}{\frac{1}{2}-\alpha'(s-m_\pi^2)}+\frac{6s}{\frac{1}{2}-\alpha'(s-2m_K^2+m_\pi^2)}\right]\times$$

$$\times (s-4m_K^2) \quad . \qquad (2.92)$$

$$S_0^1(K\bar{K}) = \beta''\alpha'\left[\frac{1}{\frac{1}{2}-\alpha'(s-m_\pi^2)}\right][-s-\frac{2}{3}\alpha'(s^2-8m_K^2s+4m_K^4)$$

$$-\frac{2}{3}\alpha'(s-4m_K^2)(s+2m_K^2)\ln 2] \qquad (2.93)$$

$$S_1^0(K\bar{K}) = \frac{\beta''}{3}\alpha'(s-4m_K^2)\left\{\left[\frac{1}{\frac{1}{2}-\alpha'(s-2m_K^2+m_\pi^2)}+\frac{1}{\frac{1}{2}-\alpha'(s-m_\pi^2)}\right]\times\right.$$

$$\times [-1-\alpha'(s-4m_K^2)]$$

$$\left.+4\alpha'\ln 2\left[\frac{(s-4m_K^2)}{\frac{1}{2}-\alpha'(s-2m_K^2+m_\pi^2)}+\frac{m_K^2}{\frac{1}{2}-\alpha'(s-m_\pi^2)}\right]\right\} \qquad (2.94)$$

$$S_1^1(K\bar{K}) = \frac{\alpha'\beta''}{\frac{1}{2}-\alpha'(s-m_\pi^2)}\left[\frac{s-4m_K^2}{3}\right](-1-\alpha'(s-4m_K^2)+4\alpha' m_K^2 \ln 2). \tag{2.95}$$

The $K\bar{K}$ scattering lengths derived from the amplitudes (2.92)-(2.95) are

$$a_0^0(K\bar{K}) = \left.\frac{S_0^0(K\bar{K})}{8\pi\sqrt{s}}\right|_{s=4m_K^2} = \frac{\beta''\alpha'}{16\pi m_K}[-4m_K^2+8\alpha' m_K^4]\times$$

$$\times \left[\frac{1}{\frac{1}{2}-\alpha'(4m_K^2)} + \frac{2}{\frac{1}{2}-\alpha'(2m_K^2)}\right].$$

To avoid the divergence of the second term in the last bracket, the more precise value of $2\alpha' m_K^2 = 0.45$ has to be used instead of 0.5. One then gets

$$a_0^0(K\bar{K}) \approx 9.28\ m_K^{-1} \tag{2.96}$$

and

$$a_0^1(K\bar{K}) = \left.\frac{S_0^1(K\bar{K})}{8\pi\sqrt{s}}\right|_{s=4m_K^2} \approx \frac{\beta''\alpha'}{4\pi m_K}[-m_K^2+2\alpha' m_K^4]\left[\frac{1}{\frac{1}{2}-\alpha'(4m_K^2)}\right]$$

$$\approx -0.6\ m_K^{-1}. \tag{2.97}$$

The p wave scattering lengths are

$$a_1^0(K\bar{K}) = \left.\frac{S_1^0(K\bar{K})}{16\pi m_K p^2}\right|_{s=4m_K^2} = -\frac{\beta''\alpha'}{12\pi m_K}\left\{\frac{2}{\frac{1}{2}-2\alpha' m_K^2} + \frac{1}{\frac{1}{2}-4\alpha' m_K^2}\right.$$

$$\left. + \frac{4\alpha' m_K^2 \ln 2}{\frac{1}{2}-4\alpha' m_K^2}\right\}$$

$$\approx 5.66\ m_K^{-3} \tag{2.98}$$

$$a_1^1(K\bar{K}) = \left.\frac{S_1^1(K\bar{K})}{16\pi m_K p^2}\right|_{s=4m_K^2} = \frac{\beta''\alpha'}{12\pi m_K}[-1+4\alpha' m_K^2 \ln 2] \frac{1}{\frac{1}{2}-4\alpha' m_K^2}$$

$$\approx -0.136\ m_K^{-3} \quad . \tag{9.99}$$

The results on the s-wave scattering lengths, eqs. (2.96) and (2.97) coincide with those of the Kawarabayashi's$^{(4)}$ calculations and are quite different from the current algebra predictions. The current algebra prediction for $K\bar{K}$ scattering lengths is not expected to be very good as the $K\bar{K}$ channel is rich in bound states and the extrapolation from the Adler zero point to the $K\bar{K}$ threshold is not expected to be very smooth. Kawarabayashi claims that the large $K\bar{K}$ s-wave scattering lengths are in good agreement with the experimental data$^{(19)}$.

Working in the t-channel, one gets the amplitude for the KK→KK process. The isospin amplitudes, as given earlier, are

$$T^0 = -\beta''(V_{\rho\phi}(u,s) - V_{\rho\phi}(s,u)) \tag{2.100}$$

$$T^1 = \beta''(V_{\rho\phi}(u,s) + V_{\rho\phi}(s,u)) \quad . \tag{2.101}$$

In the t channel centre of mass system, the Mandelstam variables s, t and u become

$$t = (p-q')^2 = (p+k')^2 = p_o^2+2p_o k_o+k_o^2 \tag{2.102}$$

$$s = (p+q)^2 = (p-k)^2 = 4m_K^2 - t + 2p^2(1-\cos\theta_t) \qquad (2.103)$$

$$u = (p-p')^2 = -2p^2(1-\cos\theta_t) \qquad (2.104)$$

where $k' = -q'$ and $k = -q$

$$\cos\theta_t = 1 + \frac{u}{2p^2} \quad \text{and} \quad 4p^2 = t - 4m_K^2 \ . \qquad (2.105)$$

At threshold $t \to 4m_K^2$, and both $1-\alpha_\rho(s)$ and $1-\alpha_\phi(u) > 0$, therefore we can write

$$V_{\rho\phi}(s,u) = (1-\alpha_\rho(s)-\alpha_\phi(u)) \int_0^\infty \frac{(1+\xi)^{\alpha_\rho(s)+\alpha_\phi(u)}}{(1+\xi)^2\ \xi^{\alpha_\phi(u)}}\, d\xi$$

$$= \alpha'(2m_K^2-s-u) \int_0^\infty \frac{(1+\xi)^{\alpha'(s+u-2m_K^2)}}{(1+\xi)\,\xi^{\frac{1}{2}+\alpha'(u-2m_K^2+m_\pi^2)}}\, d\xi . \qquad (2.106)$$

Along the same line of treatment as in the previous sections, we can project out the s and p wave amplitudes of $V_{\rho\phi}(s,u)$

$$V_{\rho\phi}(s,u)_{\ell=0} = \frac{1}{2} \int_{-1}^{1} V_{\rho\phi}(s,u)\, d\cos\theta_t$$

$$= 2\alpha'(\frac{t}{2} - s) + 4\alpha'^2[s(2m_K^2-s) - \frac{1}{3}(t-4m_K^2)(t-3s-m_K^2)]$$

$$+ \alpha'^2(t-4m_K^2)[\frac{1}{3}(t-3s+2m_K^2)] \ln 2 \qquad (2.107)$$

$$V_{\rho\phi}(s,u)_{\ell=1} = \frac{1}{2}\int_{-1}^{1} V_{\rho\phi}(s,u)\cos\theta_t d\,\cos\theta_t$$

$$= -\frac{4}{3}\alpha' p^2 - \frac{\alpha'^2}{3}(4p^2)[(4s-4m_K^2-8p^2) - 2\ln 2(s-2m_K^2-4p^2)] \; . \tag{2.108}$$

Similarly, one finds

$$V_{\rho\phi}(u,s) = \alpha'(2m_K^2-s-u)\int_0^\infty \frac{(1+\xi)^{\alpha'(s+u-2m_K^2)}}{(1+\xi)\xi^{\frac{1}{2}+\alpha'(u-m_\pi^2)}} d\xi , \tag{2.109}$$

$$V_{\rho\phi}(u,s)_{\ell=0} = \frac{1}{2}\int_{-1}^{1} V_{\rho\phi}(u,s) d\,\cos\theta_t$$

$$= \alpha'(t-2s)+4\alpha'^2[s(2m_K^2-s) - \frac{(t-4m_K^2)(t-3s-m_K^2)}{3}]$$

$$- 2\alpha'^2(\ln 2)s[t-2s] \tag{2.110}$$

and,

$$V_{\rho\phi}(u,s)_{\ell=1} = -\frac{4}{3}\alpha' p^2 - \frac{4p^2}{3}\alpha'^2[4(s-m_K^2-2p^2)-2s\ \ln 2] \; . \tag{2.111}$$

The s and p wave amplitudes finally are

$$T_0^0(KK) = \beta''\alpha'^2 \ln 2\{2s(t-2s)+\tfrac{1}{3}(t-4m_K^2)(t-3s+2m_K^2)\} \tag{2.112}$$

$$T_0^1(KK) = \beta''\alpha'\{2(t-2s)+8\alpha'[s(2m_K^2-s) - \tfrac{1}{3}(t-4m_K^2)(t-3s+2m_K^2)]$$

$$- \alpha' \ln 2[2s(t-2s) - \tfrac{1}{3}(t-4m_K^2)(t-3s+2m_K^2)]\} . \tag{2.113}$$

$$T_1^o(KK) = -\frac{16}{3}\, p^2 \alpha' \beta'' (m_K^2 + 2p^2) \ln 2 \tag{2.114}$$

$$T_1^1(KK) = -\frac{8}{3}\, p^2 \alpha' \beta'' [1 + 2\alpha'(s - m_K^2 - 2p^2)(2 - \ln 2)]. \tag{2.115}$$

From these amplitudes (2.112)-(2.115), we get the following scattering lengths

$$a_o^o(KK) = \left. \frac{T_o^o(KK)}{16\pi\sqrt{t}} \right|_{\substack{t=4m_K^2 \\ s=0}} = 0 \tag{2.116}$$

$$a_o^1(KK) = \left. \frac{T_o^1(KK)}{16\pi\sqrt{t}} \right|_{\substack{t=4m_K^2 \\ s=0}} = \frac{1}{4\pi}\, \beta'' \alpha' m_K = -0.436\ m_K^{-1} \tag{2.117}$$

$$a_1^o(KK) = \left. \frac{T_1^o(KK)}{16\pi\sqrt{t}\ p^2} \right|_{\substack{t=4m_K^2 \\ p=0}} = -\frac{\alpha'^2 \beta''}{6\pi}\, m_K \ln 2$$

$$\approx 0.0044\ m_K^{-3} \tag{2.118}$$

$$a_1^1(KK) = \left. \frac{T_1^1(KK)}{16\pi\sqrt{t}\ p^2} \right|_{\substack{t=4m_K^2 \\ p=0}} = -\frac{\alpha' \beta''}{12\pi m_K}[1 - 2\alpha' m_K (2 - \ln 2)]$$

$$\approx 0.059\ m_K^{-3}\ . \tag{2.119}$$

Since the KK scattering amplitudes do not have bound states or resonances the deviation from the current algebra results should be small. In the case of the KK s-wave scattering lengths this is true(18).

CHAPTER III

VENEZIANO MODEL OF $\pi\pi\to\pi A_1$ REACTION

Section I : General Properties of the $\pi\pi\to\pi A_1$ Scattering Amplitude

Consider the reaction

$$\pi^-(q) + \pi^+(p) \longrightarrow \pi^-(q') + A_1^+(k) \ ,$$

where q, p, q' and k refer to the particle four-momenta. The Mandelstam variables are

$$s = (p+q)^2 \quad t = (q-q')^2 \quad u = (k-q)^2 \qquad (3.1)$$

with $s+t+u = m_{A_1}^2 + 3m_\pi^2$. Because of the energy-momentum conservation there are three independent four momenta in the problem which one could choose as $(q+q')_\mu$, $(q-q')_\mu$ and, say, p_μ. The gauge condition $\eta.k = 0$ (where η_μ is the polarization four-vector of the A_1^+) implies that $\eta.p = \eta.(k+q'-q) = \eta.(q'-q)$. Thus two invariant amplitudes, called A(s,t) and B(s,t) in the following, serve to define the complete amplitude. One can therefore write the amplitude for $\pi\pi\to\pi A_1$ as$^{(7)}$

$$T_{\pi\pi,\pi A_1}(s,t) = A(s,t)(q-q').\eta + B(s,t)(q+q').\eta \ . \qquad (3.2)$$

Under the exchange of s and t, we see from eq. (2) that

$$A(s,t) \longleftrightarrow \frac{1}{2}[3B(t,s) - A(t,s)]$$

(3.3)

$$B(s,t) \longleftrightarrow \frac{1}{2}[A(t,s) + B(t,s)] \; .$$

The amplitude $T_{\pi\pi,\pi A_1}(s,t)$ in eq. (2) can alternatively be written as

$$T_{\pi\pi,\pi A_1}(s,t) = [A(s,t) + B(s,t)](-p.\eta) + 2B(s,t)q'.\eta \; . \qquad (3.4)$$

In the s channel, the contribution of the ρ pole to the amplitude (Fig. 1a) is given by

$$g_{\rho\pi\pi}(K-2p).\varepsilon_\rho \frac{1}{s-m_\rho^2} [G_s\varepsilon_\rho.\eta + 2G_D(K.\eta)(k.\varepsilon_\rho)] \qquad (3.5)$$

where K=p+q is the four momentum transferred, ε_ρ is the polarization of the ρ meson and the coupling constants are defined as follows: The $\pi^-(q) + \pi^+(p) \longrightarrow \rho^o(K)$ vertex is defined as,

$$V_{\rho\pi\pi} = g_{\rho\pi\pi}(q-p).\varepsilon_\rho \quad , \qquad (3.6)$$

and the

$$\rho^o(K) \longrightarrow \pi^-(q') + A_1^+(k)$$

vertex as,

$$V_{\rho\pi A_1} = G_s\varepsilon_\rho.\eta + 2G_D(K.\eta)(k.\varepsilon_\rho) \quad . \qquad (3.7)$$

By a rearrangement of the parameters in eq. (3.5), we can write it as

$$g_{\rho\pi\pi}\{2G_S p.\eta + (q'.\eta)[G_S - G_D(m_{A_1}^2 + 3m_\pi^2 - s - 2t]\} \frac{1}{s-m_\rho^2} . \tag{3.8}$$

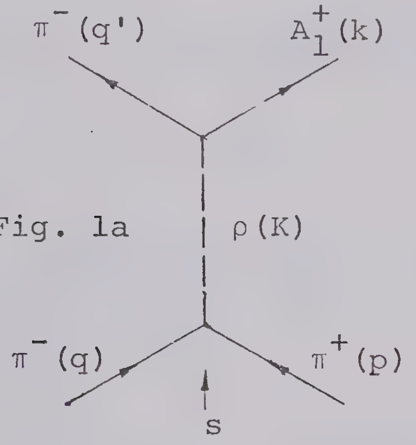


Fig. 1a

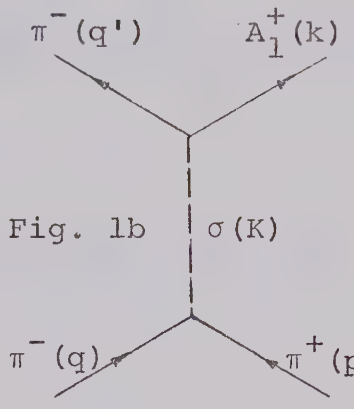


Fig. 1b

The contribution of the σ (the 0^+ ρ-daughter) to the amplitude (Fig. 1b) is given by

$$g_{\sigma\pi\pi} g_{\sigma\pi A_1} (2K-k).\eta \frac{1}{s-m_\rho^2} = g_{\sigma\pi\pi} g_{\sigma\pi A_1} 2q'.\eta \frac{1}{s-m_\rho^2} . \tag{3.9}$$

The coupling constants in (3.9) are defined as follows. The $\pi^-(q) + \pi^+(p) \longrightarrow \sigma(K)$ vertex is given by

$$V_{\sigma\pi\pi} = g_{\sigma\pi\pi} , \tag{3.10}$$

and the $\sigma(K) \rightarrow \pi^-(q') + A_1^+(k)$ by

$$V_{\sigma\pi A_1} = g_{\sigma\pi A_1} (K+q').\eta . \tag{3.11}$$

Compare eqs. (3.8) and (3.9) to (3.4), one finds that

$$B(s,t) = -\frac{1}{2} g_{\rho\pi\pi}[G_S-G_D(m_{A_1}^2+3m_\pi^2-s-t)]\frac{1}{s-m_\rho^2}+g_{\sigma\pi\pi}g_{\sigma\pi A_1}\frac{1}{s-m_\rho^2} \tag{3.12}$$

$$A(s,t) = -\frac{1}{2} g_{\rho\pi\pi}[3G_S+G_D(m_{A_1}^2+3m_\pi^2-s-t)]\frac{1}{s-m_\rho^2}+g_{\sigma\pi\pi}g_{\sigma\pi A_1}\frac{1}{s-m_\rho^2}\,. \tag{3.13}$$

Similarly, one can obtain the ρ-pole and σ-pole contributions to the amplitude in the t channel (Fig. 2a and Fig. 2b).

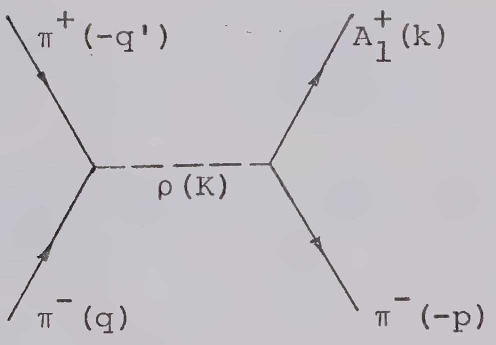


Fig. 2a

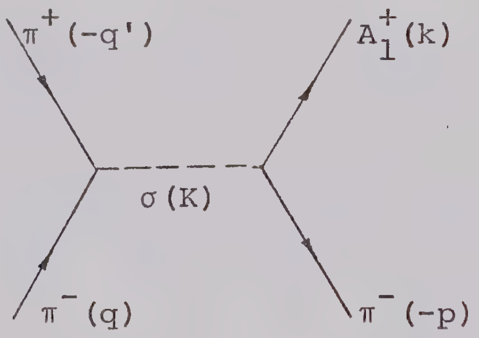


Fig. 2b

$$\begin{aligned}T_{\rho\text{-contribution}} &= g_{\rho\pi\pi}(K+2q')\cdot\varepsilon_\rho\,\frac{1}{t-m_\rho^2}[G_S\varepsilon_\rho\cdot\eta+2G_D(K\cdot\eta)(k\cdot\varepsilon_\rho)]\\ &=-g_{\rho\pi\pi}\{2q'\cdot\eta G_S-p\cdot\eta[G_S-G_D(m_{A_1}^2+3m_\pi^2-t-2s)]\}\frac{1}{t-m_\rho^2},\end{aligned} \tag{3.14}$$

$$T_{\sigma\text{-contribution}}=-2g_{\sigma\pi\pi}g_{\sigma\pi A_1}\,p\cdot\eta\,\frac{1}{t-m_\rho^2}\,. \tag{3.15}$$

From eqs. (3.14), (3.15) and (3.4), one gets

$$B(s,t) = -g_{\rho\pi\pi} G_s / t - m_\rho^2 \quad , \tag{3.16}$$

$$A(s,t) = g_{\rho\pi\pi} G_D (m_{A_1}^2 + 3m_\pi^2 - t - 2s) \frac{1}{t-m_\rho^2} + 2 g_{\sigma\pi\pi} g_{\sigma\pi A_1} \frac{1}{t-m_\rho^2} \quad . \tag{3.17}$$

Carrying out the partial wave expansion of the invariant amplitudes A(s,t) and B(s,t), in the t channel, one obtains for the two helicity states $\lambda = 1$ and $\lambda = 0$ (7) (20).

$$\sqrt{2} q_t B(s,t) = \sum_J \frac{2J+1}{[J(J+1)]^{\frac{1}{2}}} <10|T_J|00> P_J'(\cos\theta_t) \quad , \tag{3.18}$$

$$\frac{2}{m_{A_1}} [\omega_t k_t A(s,t) - q_t \varepsilon_t B(s,t) \cos\theta_t] = \sum_J (2J+1) <00|T_J|00> P_J(\cos\theta_t) , \tag{3.19}$$

where

$$k_t = [t^2 - 2(m_{A_1}^2 + m_\pi^2)t + (m_{A_1}^2 - m_\pi^2)^2]^{\frac{1}{2}} / 2\sqrt{t} \quad ,$$

$$q_t = \frac{1}{2}(t - 4m_\pi^2)^{\frac{1}{2}} \quad , \quad \varepsilon_t = (k_t^2 + m_{A_1}^2)^{\frac{1}{2}} \quad \text{and} \quad \omega_t = (q_t^2 + m_\pi^2)^{\frac{1}{2}} \quad .$$

In the s channel, the partial wave expansions for A(s,t) and B(s,t) in the helicity 1 and 0 states are,

$$\frac{1}{\sqrt{2}} q_s [A(s,t) + B(s,t)] = \sum_J \frac{2J+1}{[J(J+1)]^{\frac{1}{2}}} <10|T_J|00> P_J'(\cos\theta_s) \quad , \tag{3.20}$$

$$\frac{1}{m_{A_1}}\,[\omega_s k_s(-A+3B)-q_s\varepsilon_s(A+B)\cos\theta_s] = \sum_J (2J+1)\langle 00|T_J|00\rangle P_J(\cos\theta_s).$$

As $s\to\infty$, $P_J'(\cos\theta_t)\propto s^{\alpha(t)-1}$ and $P_J(\cos\theta_t)\propto s^{\alpha(t)}$, while as $t\to\infty$, $P_J'(\cos\theta_s)\propto t^{\alpha(s)-1}$ and $P_J(\cos\theta_s)\propto t^{\alpha(s)}$. Therefore, from eqs. (3.12), (3.13), (3.16), (3.17) and the above equations, one obtains the asymptotic behaviours for $A(s,t)$ and $B(s,t)$

$$A(s,t)\sim s^{\alpha(t)},\quad B(s,t)\sim s^{\alpha(t)-1}\quad \text{for } t \text{ fixed and } s\to\infty\,,$$

$$A(s,t)\sim t^{\alpha(s)},\quad B(s,t)\sim t^{\alpha(s)}\quad \text{for } s \text{ fixed and } t\to\infty\,. \tag{3.22}$$

According to the Adler's consistency condition$^{(5)}$, the amplitude for the $\pi\pi\to\pi A_1$ reaction vanishes as one of the pion four momentum goes to zero. Hence, from eq. (3.2) one gets the following:

(i) When $q'=0$, $T(s,t)=0$ gives

$$A(s,t)+B(s,t) = 0 \quad \text{at} \quad s=m_{A_1}^2\,,\quad t=m_\pi^2\,; \tag{3.23}$$

(ii) When $q=0$, $T(s,t)=0$ gives

$$A(s,t)-B(s,t) = 0 \quad \text{at} \quad s=m_\pi^2\,,\quad t=m_\pi^2\,; \tag{3.24}$$

(iii) When $p=0$, $T(s,t)=0$ gives

$$B(s,t) = 0 \quad \text{at} \quad s=m_\pi^2\,,\quad t=m_{A_1}^2\,. \tag{3.25}$$

Section II : Veneziano Model for $\pi\pi\to\pi A_1$ Amplitude

One can write the Veneziano formula for amplitudes A(s,t) and B(s,t) as

$$B(s,t) = (\gamma_1 + \gamma_2 t)\,\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))}\,, \qquad (3.26)$$

$$A(s,t) = (\gamma_1 + 2\gamma_2 s - \gamma_2 t)\,\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} \qquad (3.27)$$

in which B(s,t) is the same formula as given by Riazuddin and Fayyazuddin(7); and A(s,t) is so written that $T_{\pi\pi,\pi A_1}(s,t)$, eq. (3.2), is equivalent to that given by Rosner and Surra(21). It can be shown that $\gamma_1 = \frac{\gamma'}{2}$ and $\gamma_2 = -\alpha'\gamma$, where γ and γ' are the parameters introduced in eq. (2) in reference (21). We also see from (3.26) and (3.27) that A(s,t) and B(s,t) satisfy the crossing condition (3.3) and that both amplitudes give the correct Regge behaviour for large t or s.

Generally the Adler zeros are enforced via a pole of the gamma function in the denominators of A(s,t) and B(s,t)(7). Such a constraint gives a mass relation between m_{A_1}, m_ρ and m_π, i.e.,

$$m_{A_1}^2 + m_\pi^2 = 2m_\rho^2 \quad . \qquad (3.28)$$

This relation is also obtained from Weinberg sum-rules based on chiral symmetry. This has been the usual procedure pursued in all the published works.

Notice however that both A(s,t) and B(s,t) have gamma functions multiplied by a polynomial in s and t. There is therefore an alternative way to enforce the Adler zero namely by demanding that these polynomials vanish at the kinematical points required by the Adler zero. We have followed this procedure to see what consequences follow from it and whether there is any evidence that this is not a consistent procedure. Thus, from (3.23)-(3.25), we have

(i) A zero for B(s,t) at $t=m_{A_1}^2$, $s=m_\pi^2$ provided that

$$\gamma_1 + \gamma_2 m_{A_1}^2 = 0 \; ; \tag{3.29}$$

(ii) At $s=m_{A_1}^2$, $t=m_\pi^2$, T(s,t)=0 requires

$$A(s,t) + B(s,t) = 0 \; ,$$

$$\text{or} \quad 2\gamma_1 + 2\gamma_2 \, s = 2(\gamma_1 + \gamma_2 m_{A_1}^2) = 0 \; ;$$

(iii) At $s=m_\pi^2$, $t=m_\pi^2$, $A(s,t)-B(s,t) = 0$,

$$\text{or} \quad 2(\gamma_2 s - \gamma_2 t) = 0 \; .$$

Hence all the zeros can be secured if (3.29) holds, i.e. $\gamma_1 = \gamma_2 m_{A_1}^2$. From eq. (3.26), we see that the residue of B(s,t) at the pole $s=m_\rho^2$ is

$$-(\gamma_1 + \gamma_2 t)\,\frac{1}{\alpha'} \quad , \tag{3.30}$$

where α' is the slope of the ρ-trajectory, and $\alpha' = [2(m_\rho^2 - m_\pi^2)]^{-1}$. Comparing eqs. (3.12) and (3.30), one gets

$$\frac{\gamma_2}{\alpha'} = G_D g_{\rho\pi\pi} \quad , \tag{3.31}$$

$$\frac{\gamma_1}{\alpha'} = \frac{1}{2} g_{\rho\pi\pi}[G_S - G_D(m_{A_1}^2 - m_\rho^2 + 3m_\pi^2)] - g_{\sigma\pi\pi} g_{\sigma\pi A_1} \ . \tag{3.32}$$

At $t = m_\rho^2$, the residue of $B(s,t)$ is

$$-(\gamma_1 + \gamma_2 m_\rho^2)/\alpha' \ . \tag{3.33}$$

From eqs. (3.16) and (3.33), one obtains

$$(\gamma_1 + \gamma_2 m_\rho^2)/\alpha' = g_{\rho\pi\pi} G_S \ . \tag{3.34}$$

It follows from eqs. (3.31), (3.32) and (3.34) that

$$g_{\sigma\pi\pi} g_{\sigma\pi A_1} = -\frac{1}{2} g_{\rho\pi\pi}[G_S + G_D(m_{A_1}^2 - 3m_\rho^2 + 3m_\pi^2)] \ . \tag{3.35}$$

If we impose the condition that $\gamma_1 = -\gamma_2 m_{A_1}^2$, then eq. (3.34) gives

$$\gamma_2(m_{A_1}^2 - m_\rho^2)/\alpha' = -g_{\rho\pi\pi} G_S \quad , \tag{3.36}$$

which when combine with eq. (3.31) produces

$$\gamma = G_S/G_D = -(m_{A_1}^2 - m_\rho^2) \; . \tag{3.37}$$

Using the mass formula $m_{A_1}^2 + m_\pi^2 = 2m_\rho^2$ which we shall use as an empirical relation, one gets

$$\gamma = G_S/G_D = -(m_\rho^2 + m_\pi^2) \; . \tag{3.38}$$

From eqs. (3.31), (3.32) and the condition $\gamma_1 = -\gamma_2 m_{A_1}^2$, one finds that

$$g_{\sigma\pi\pi} g_{\sigma\pi A_1} = \frac{1}{2}\, g_{\rho\pi\pi} [G_S + G_D (m_{A_1}^2 + m_\rho^2 - 3m_\pi^2)] \; , \tag{3.39}$$

with $G_S/G_D = -(m_{A_1}^2 - m_\rho^2)$, both (3.35) and (3.39) give

$$g_{\sigma\pi\pi} g_{\sigma\pi A_1} = \frac{1}{2}\, g_{\rho\pi\pi} G_D [2m_\rho^2 - 3m_\pi^2] \; . \tag{3.40}$$

If we use the relation $g_{\sigma\pi\pi}^2 = m_\rho^2 g_{\rho\pi\pi}^2$ (2), which results from the Veneziano model of the ππ→ππ scattering, we can show from (3.40) that

$$g_{\sigma\pi A_1} = \frac{1}{2m_\rho} G_D (2m_\rho^2 - 3m_\pi^2) \quad , \tag{3.41}$$

or

$$g_{\sigma\pi A_1} \approx G_D m_\rho \qquad \text{for} \quad m_\pi = 0 \; . \tag{3.42}$$

If the σ-contribution were absent in eq. (3.35), one would obtain the result of reference (7)*,

---

* The G_D of reference (7) has opposite sign of ours because of the difference in metric matrix used.

$$\gamma = G_s/G_D = -(m_{A_1}^2 - 3m_\rho^2 + 3m_\pi^2) ,$$

or

$$\gamma \approx m_\rho^2 \quad \text{for} \quad m_\pi^2 \approx 0 \quad \text{and} \quad m_{A_1}^2 \approx 2m_\rho^2 . \qquad (3.43)$$

Note that the result here is opposite in sign to (3.38) which is

$$\gamma \approx -m_\rho^2 \quad \text{for} \quad m_\pi^2 \approx 0 .$$

Rosner and Surra$^{(21)}$, using the experimental result of the SLAC group, estimated that γ may take the values

(i) $-0.1\ m_\rho^2 \leq \gamma_- \leq -0.06\ m_\rho^2$,

(ii) $0.16\ m_\rho^2 \leq \gamma_+ \leq 0.44\ m_\rho^2$.

Section III : Decay Rates for $A_1 \to \pi\sigma$ and $A_1 \to \rho\pi$ Reactions

Using the value $\gamma \approx -m_\rho^2$, the decay widths for $A_1 \to \sigma\pi$ and $A_1 \to \rho\pi$ are calculated and listed below

$$\Gamma(A_1 \to \sigma\pi) = \frac{g_{\sigma\pi A_1}^2}{12\pi} \frac{m_\rho}{16\sqrt{2}}$$

$$\Gamma(A_1 \to \rho\pi)_{\lambda=0} = \frac{G_D}{4\pi} \frac{(3\gamma + m_\rho^2)^2}{96\sqrt{2}\ m_\rho} = \frac{G_D^2}{4\pi} \frac{m_\rho^3}{24\sqrt{2}} ,$$

$$\Gamma(A_1 \to \rho\pi)_{\lambda=1} = \frac{G_D}{4\pi} \frac{\gamma^2}{6\sqrt{2}\ m_\rho} = \frac{G_D}{4\pi} \frac{m_\rho^3}{6\sqrt{2}} ,$$

$$\Gamma(A_1 \rightarrow \rho\pi)_{\text{total}} = \frac{G_D^2}{4\pi} \frac{5m_\rho^3}{24\sqrt{2}}$$

where λ denotes the helicity state. Using $g_{\sigma\pi A_1} \approx G_D m_\rho$, one can estimate the ratio of these partial widths.

$$\frac{\Gamma(A_1 \rightarrow \rho\pi)_{\lambda=0}}{\Gamma(A_1 \rightarrow \rho\pi)_{\lambda=1}} = \frac{1}{4} \quad ,$$

$$\frac{\Gamma(A_1 \rightarrow \sigma\pi)}{\Gamma(A_1 \rightarrow \rho\pi)_{\lambda=1}} = \frac{1}{8} \quad , \tag{3.44}$$

$$\frac{\Gamma(A_1 \rightarrow \sigma\pi)}{\Gamma(A_1 \rightarrow \rho\pi)_{\text{total}}} = \frac{1}{10} \quad .$$

In reference (7), three sets of results were given:

(i) When $g_{\sigma\pi A_1} = 0$, then $\gamma = m_\rho^2$ and one gets,

$$\Gamma(A_1 \rightarrow \sigma\pi) = 0 \quad ,$$

and
$$\frac{\Gamma(A_1 \rightarrow \rho\pi)_{\lambda=0}}{\Gamma(A_1 \rightarrow \rho\pi)_{\lambda=1}} \approx 1 \quad .$$

(ii) If $G_D = 0$, then $g_{\sigma\pi A_1} = G_s/2m_\rho$ and one gets,

$$\Gamma(A_1 \rightarrow \rho\pi)_{\lambda=0} = 0$$

and
$$\frac{\Gamma(A_1 \rightarrow \sigma\pi)}{\Gamma(A_1 \rightarrow \rho\pi)_{\lambda=1}} \approx \frac{1}{32} \quad .$$

(iii) If $G_s = 0$, then $g_{\sigma\pi A_1} = \frac{1}{2} m_\rho G_D$ and this gives

$$\Gamma(A_1 \to \rho\pi)_{\lambda=1} = 0 ,$$

$$\frac{\Gamma(A_1 \to \sigma\pi)}{\Gamma(A_1 \to \rho\pi)_{\lambda=0}} \approx \frac{1}{2} .$$

The results (3.44) are quite different from those given in the three cases and the relation $g_{\sigma\pi A_1} = m_\rho G_D$, we obtain, is twice of that given in case (iii).

CHAPTER IV

VENEZIANO MODEL OF $K_2^o \to \pi^+\pi^-\gamma$ DECAY

Section I : Introduction

In the pole model prescription for the decay processes, the initial particle is envisaged as converting weakly (or electromagnetically) into an intermediate state which subsequently decays into the final state through strong interaction. Since the successful application of Veneziano model(1) to the $\pi\pi$ scattering and spectra in 3π final state process (2)(3), several authors have used the pole model, with the strong interaction part of the decay amplitude described by the Veneziano model, to calculate the $K\to 3\pi$, $\eta\to 3\pi$ (9)(10), $K^\pm\to\pi^\pm\pi^o\gamma$ (22) and $K_2^o\to\pi^+\pi^-\gamma$ (12) decays.

Sutherland(9) and Jacob et al.(10) considered the $K\to 3\pi$ and $\eta\to 3\pi$ decays in the pole model and showed that the contribution from Fig. 1 dominates that from Fig. 2.

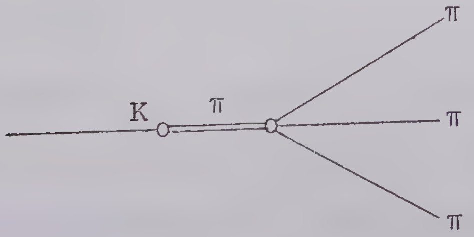


Fig. 1

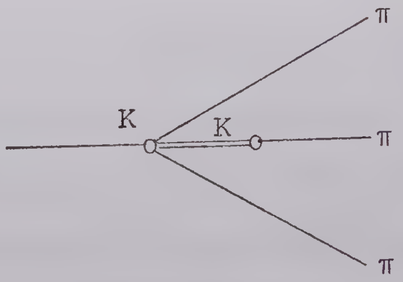


Fig. 2

In the works of references (9) and (10), the transition amplitude $<\pi|H_{wk}|K>$ for the K meson going into the intermediate pion state is calculated in terms of $<\pi\pi(0)|H_{wk}|K>$ using the current algebra. Neglecting all off mass shell effects they obtained an amplitude for $K\rightarrow 3\pi$ which was about 6 times larger than the experimental result. Therefore, they concluded that large off mass corrections were needed for the weak vertex and/or strong interaction part.

Rockmore(12) used the pion pole model for $K_2^o\rightarrow\pi^+\pi^-\gamma$ in which the K meson converts to a pion weakly through a phenomenological mixing of K and η mesons with the pion (Fig. 3). With the uncertainty in the sign of $\eta-\pi^o$ mixing relative to the $K-\pi$ mixing and also in the sign introduced by a model of PVV coupling, he calculated the branching ratio $\Gamma(K_2^o\rightarrow\pi^+\pi^-\gamma)/\Gamma(K_2^o\rightarrow all)$ for all the possible sign combinations. All the results lie within the experimental upper limit of 4×10^{-4} (23).

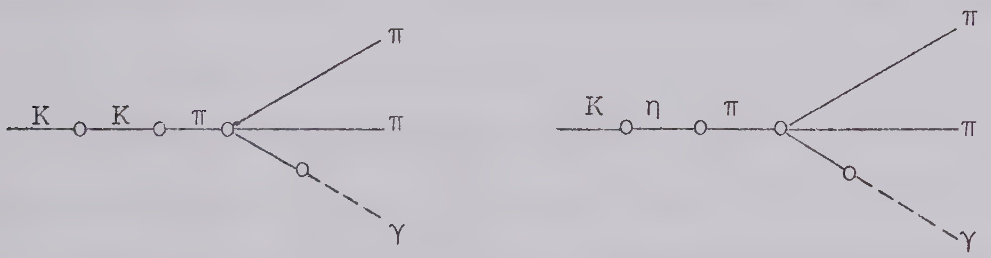


Fig. 3

In the paper of Dass and Kamal(22) on the magnetic radiation in $K^\pm\rightarrow\pi^\pm\pi^o\gamma$ decays, the intermediate state was

taken as a $J^P = 0^-$ π-like meson. They estimated the amplitude for the transition of K meson to the π-like 0^- intermediate meson (π') by relating the $K^{\pm}\rightarrow\pi^{\pm}\pi^{o}\gamma$ amplitude to the τ decay ($K\rightarrow 3\pi$) amplitude whose size is known experimentally. The function ψ defined in their paper

$$\psi = \frac{<\pi'|H_{wk}|K>}{m_K^2 - m_{\pi'}^2} g_{\rho\pi'\pi}$$

embraces all the off-mass shell factors and hence takes care of the off-mass shell effects in the K-π' transition.

In this chapter the $K_2^o\rightarrow\pi^+\pi^-\gamma$ decay will be treated along the same lines as in reference (22), that is, the intermediate state is taken as an off-mass shell pion (or 0^- π-like meson) rather than the on-mass shell pion.

The transition amplitude for $K_2^o\rightarrow\pi'^o$, $<\pi'^o|H_{wk}|K_2^o>$, is evaluated in three different ways in Section II and the decay rate for $K_2^o\rightarrow\pi^+\pi^-\gamma$ is calculated in Section III of this chapter.

As a test of the validity of the transition amplitude $<\pi'^o|H_{wk}|K_2^o>$ the pole model of the $K_2^o\rightarrow\gamma\gamma$ decay is investigated in Section IV. There, we make use of the results for $<\pi'^o|H_{wk}|K_2^o>$ amplitude obtained in Section II and consider the decay of $\pi'^o\rightarrow\gamma\gamma$, according to the vector meson dominance model, going via the $\pi'^o\rightarrow\rho\omega$ or $\pi'^o\rightarrow\rho\gamma$ processes with ρ and ω meson coupling to the final

state photons (Fig. 4 and Fig. 5).

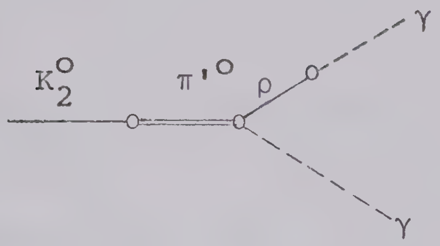


Fig. 4

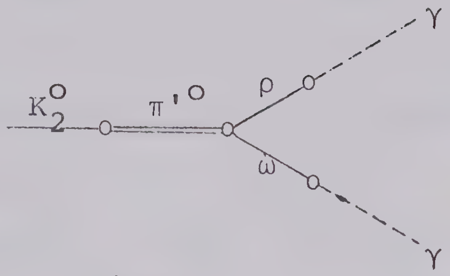


Fig. 5

Section II : Transition Amplitude for $K_2^0\to\pi^+\pi^-\gamma$ Decay

To order e the decay $K_2^0\to\pi^+\pi^-\gamma$ will proceed via the 'direct process' (Fig. 6) and the 'innerbremstrahlung process' (Fig. 7).

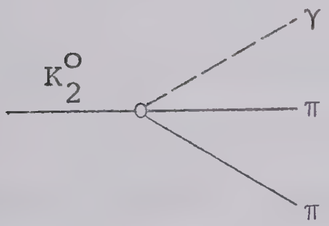


Fig. 6

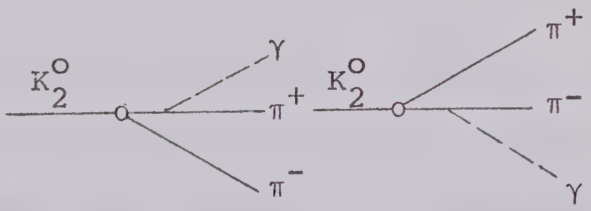


Fig. 7

If we assume cp invariance, then the processes of Fig. 7 would not be permitted. In the pole model we shall break up the vertex of the direct process (Fig. 6) in the manner illustrated in Fig. 8.

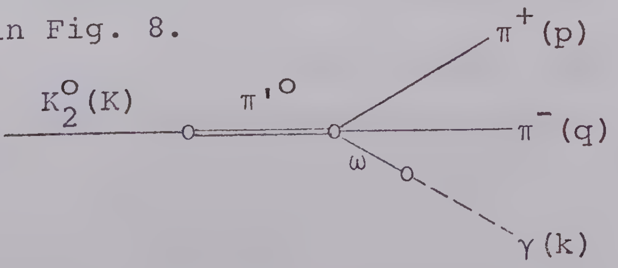


Fig. 8

If the $\Delta I = {}^{1}/2$ rule holds, the intermediate state should be either a cp = -1 and I = 0 meson (η-like meson) or a cp = -1 and I = 1 meson (π -like meson). In the previous calculation of Rockmore$^{(12)}$ there are too many unknown parameters. We choose to restrict ourselves to the discussion of the π-like meson as the only intermediate pole in the transition and neglect the η pole contribution (and the π-η mixing effect). This model will then have no free parameter and our calculation would serve to determine the adequacy or otherwise of such a model.

The part of amplitude describing the transition of $K_2^o \to \pi'^o$ is written as

$$<\pi'^{o}|H_{wk}|K_2^{o}>/(m_K^2 - m_{\pi'}^2) \tag{4.1}$$

where $m_{\pi'}$ is the effective mass of the π-like meson π'.

In the strong interaction part, making use of the vector dominance model, the π-like meson (π') is envisaged as decaying into $\pi^+\pi^-V$ state where V is a vector meson (ρ, ϕ or ω)which couples to the photon. With G parity equal to -1 for the π-like meson, the final state $\pi^+\pi^-V$ is expected to be in G = -1 state, because G parity is conserved in strong interaction. The G = +1 ρ meson cannot participate the reaction, since the G parity of the $\pi\pi\rho$ system will be +1. In the quark model with ideal nonet mixing, ϕ is made up of strange quarks only ($\phi = -\lambda\bar{\lambda}$)

while the π-like meson is made of p and n quark-antiquark combination, therefore ϕ would not be expected to couple to a 3π system. ω meson, with $G = -1$, is the only possible candidate to fit in the picture.

If π' were an on-mass shell pion, then we could apply the Veneziano function to describe the strong interaction part and write the amplitude$^{(1)}$

$$A(\pi\omega\to\pi\pi) = \beta_{\omega\pi,\pi\pi}[B_{st} + B_{su} + B_{tu}] ,$$

with

$$B_{xy} = \frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(2-\alpha(x) - \alpha(y))} , \tag{4.2}$$

where $\beta_{\omega\pi,\pi\pi}$ is the strength of the Veneziano amplitude, $\alpha(x)$ is the ρ trajectory and s, t and u are Mandelstam variables

$$s = (K-k)^2 , \quad t = (p+k)^2 , \quad u = (q+k)^2 . \tag{4.3}$$

K, k, p and q are the four momenta of the K meson, photon, π^+ and π^- respectively. The amplitude $A(\pi\omega\to\pi\pi)$ is symmetrical under the exchange of s, t and u. For an off-mass shell π' meson, the amplitude $A(\pi'\omega\to\pi\pi)$ is only symmetrical in t and u. The general form for $A(\pi'\omega\to\pi\pi)$ should be written as

$$A(\pi'\omega\to\pi\pi) = \beta_{\pi'\omega,\pi\pi}[B_{st} + B_{su} + \xi B_{tu}] \quad ,$$

or

$$A(\pi'\omega\to\pi\pi) = \beta_{\pi'\omega,\pi\pi}[B_{st} + B_{su} + B_{tu} + (\xi-1)B_{tu}] \quad , \qquad (4.4)$$

where the parameter ξ measures the deviation of $A(\pi'\omega\to\pi\pi)$ from $A(\pi\omega\to\pi\pi)$. Normalizing the amplitude for $\pi'\omega\to\pi\pi$ at the ρ pole $\beta_{\omega\pi',\pi\pi}$ can be identified as

$$\beta_{\omega\pi',\pi\pi} = \alpha' g_{\rho\pi'\omega} g_{\rho\pi\pi} \quad , \qquad (4.5)$$

where α' is the slope of the ρ trajectory ($\alpha' = 0.89$ Gev^{-2}). We shall assume $g_{\rho\pi'\omega}$ to be the same as $g_{\rho\pi\omega}$ where π is the on-mass shell pion. $g_{\rho\pi\pi}$ is defined with all the particles on their mass shells and is related to the experimental ρ-width.

Finally, it follows from the vector meson dominance model that the strength of the coupling of the ω meson to the photon is given by$^{(24)}$

$$\frac{e m_\omega^2 \sin\theta_Y}{2 f_Y} \quad , \qquad (4.6)$$

in which θ_Y and f_Y are defined, in the scheme of ω-ϕ mixing, by

$$f_Y Y = \int d^3x[\cos\theta_Y J_o^{(\phi)} - \sin\theta_Y J_o^{(\omega)}] \quad ,$$

where Y is the hypercharge and $J_o^{(\phi)}$, $J_o^{(\omega)}$ are the sources of ω and ϕ meson. The total amplitude for $K_2^o \to \pi^+\pi^-\gamma$ is given by

$$A(K_2^o \to \pi^+\pi^-\gamma) = \frac{<\pi'^o|H_{wk}|K_2^o>}{m_K^2 - m_{\pi'}^2} \beta_{\omega\pi',\pi\pi} [B_{st} + B_{su} + \xi B_{tu}] \times$$

$$\times \frac{e \sin\theta_Y}{2f_Y} \varepsilon_{\mu\nu\sigma\lambda} p^\mu q^\nu k^\sigma \varepsilon^\lambda \quad , \qquad (4.7)$$

where ε is the polarization four-vector of the photon.

In eq. (4.7) all the quantities are known or can be calculated except $<\pi'^o|H_{wk}|K_2^o>/(m_K^2-m_{\pi'}^2)$. In the following we describe different models to estimate the size of this quantity.

(i) If π'^o is an on-mass shell pion, then current algebra can be used to relate the transition amplitude $<\pi^o|H_{wk}|K_2^o>$ to the amplitude $<\pi^o\pi^o(0)|H_{wk}|K_1^o>$, in which $\pi^o(0)$ is a soft pion[(9)(10)], as

$$<\pi^o|H_{wk}|K_2^o> = 2 f_\pi <\pi^o\pi^o(0)|H_{wk}|K_1^o> \quad , \qquad (4.8)$$

where f_π is defined by the PCAC hypothesis relating the divergence of the axial vector current to the pion field, $\partial_\mu J_\mu = f_\pi m_\pi^2 \phi$, and $f_\pi \approx 91$ Mev. If we take $<\pi^o\pi^o(0)|H_{wk}|K_1^o>$ to be the physical amplitude which is[(25)] 0.39×10^{-6} Gev, we obtain

$$<\pi'^o|H_{wk}|K_2^o>/(m_K^2 - m_{\pi'}^2) = 3.05 \times 10^{-7} \ . \qquad (4.9)$$

Let us look at the pole model of $K_2^o \to \pi^+\pi^-\pi^o$ decay (Fig. 1) whose amplitude is given by

$$A(K_2^o \to \pi^+\pi^-\pi^o) = \frac{<\pi^o|H_{wk}|K_2^o>}{m_K^2 - m_{\pi'}^2} \frac{\beta}{2} [V_{tu} - V_{st} - V_{su}] \ , \qquad (4.10)$$

where

$$V_{xy} = \frac{\Gamma(1-\alpha(x))\Gamma(1-\alpha(y))}{\Gamma(1-\alpha(x)-\alpha(y))} \ ,$$

and β, when normlized to the residue at the ρ-pole in $\pi\pi$ scattering, is given by $\beta = 2g_{\rho\pi\pi}g_{\rho\pi\pi}$. If the value of (4.9) is put in (4.10), we obtain $A(K_2^o \to \pi^+\pi^-\pi^o)$ at the centre of the Dalitz plot $(s = t = u = \frac{m_K^2}{3} + m_\pi^2 \approx 0.1 \text{ Gev}^2)$ to be 5.5×10^{-6} which is about six times larger than the experimental value(26) $(0.89 \pm 0.03) \times 10^{-6}$. Hence, the off-mass shell effect is expected to be large.

(ii) We may reverse the situation and take the amplitude $A(K_2^o \to \pi^+\pi^-\pi^o)$ as a known quantity and calculate the size of $<\pi'^o|H_{wk}|K_2^o>/(m_K^2 - m_{\pi'}^2)$ in which π'^o is an off-mass shell pion, we then get

$$<\pi'^o|H_{wk}|K_2^o>/(m_K^2 - m_{\pi'}^2) = 0.475 \times 10^{-7} \ .$$

(iii) From the $\Delta I = {}^1/2$ rule, one has

$$<\pi'^+|H_{wk}|K^+> = \sqrt{2}<\pi'^o|H_{wk}|K^o> = <\pi'^o|H_{wk}|K_2^o> \ . \qquad (4.12)$$

In the pole model, the amplitude for $K^+ \to \pi^+\pi^+\pi^-$ is given by

$$A(K^+ \to \pi^+\pi^+\pi^-) = \frac{\langle \pi'^+ | H_{wk} | K^+ \rangle}{m_K^2 - m_{\pi'}^2} \beta V_{tu} \quad , \qquad (4.13)$$

where the Veneziano function V_{tu} is defined in (4.10). At the centre of the Dalitz plot (4.13) becomes

$$A(K^+ \to \pi^+\pi^+\pi^-) = 0.72\,\beta \langle \pi'^+ | H_{wk} | K^+ \rangle / (m_K^2 - m_{\pi'}^2) \quad . \qquad (4.14)$$

With the experimental value of[(25)] $A(K^+ \to \pi^+\pi^+\pi^-) = 2 \times (0.96 \pm 0.02) \times 10^{-6}$, it follows from (4.12) and (4.14) that

$$\langle \pi'^0 | H_{wk} | K_2^0 \rangle / (m_K^2 - m_{\pi'}^2) = 0.513 \times 10^{-7} \quad , \qquad (4.15)$$

which is consistent with the result in (ii). Thus in the following discussion the value of $\langle \pi'^0 | H_{wk} | K_2^0 \rangle / (m_K^2 - m_{\pi'}^2)$ is taken to be 0.5×10^{-7}.

From eq. (4.7) the total amplitude can be written as

$$A(K_2^0 \to \pi^+\pi^-\gamma) = 0.5 \times 10^{-7}\, \alpha' g_{\rho\pi\omega} g_{\rho\pi\pi} \left(\frac{\sin\theta_Y}{2f_Y}\right) [B_{st} + B_{su} + \xi B_{tu}] \times$$
$$\times\, \varepsilon_{\mu\nu\sigma\lambda}\, p^\mu q^\nu k^\sigma \varepsilon^\lambda \quad . \qquad (4.16)$$

The ratio of the coupling strength of ω and γ to that of ρ and γ is[(24)]

$$\frac{1}{g^2_{\rho\pi\pi}} : \frac{\sin^2\theta_Y}{4f_Y^2} = 9.00 : 0.65 ,$$

or

$$\frac{\sin\theta_Y}{2f_Y} = \sqrt{\frac{0.65}{9}}\ \frac{1}{g_{\rho\pi\pi}} . \qquad (4.17)$$

After all the values have been introduced in (4.14), one obtains

$$A(K_2^o\to\pi^+\pi^-\gamma) = 0.579\times10^{-7}[B_{st}+B_{su}+\xi B_{tu}]\varepsilon_{\mu\nu\sigma\lambda}p^\mu q^\nu k^\sigma \varepsilon^\lambda . \qquad (4.18)$$

Section III : Decay Rate for $K_2^o\to\pi^+\pi^-\gamma$

The decay rate of $K_2^o\to\pi^+\pi^-\gamma$ is given by

$$\frac{1}{\tau} = (2\pi)^4 \frac{1}{2K_o} \int|A|^2 \frac{d^3p}{(2\pi)^3\,2p_o}\ \frac{d^3q}{(2\pi)^3\,2q_o}\ \frac{d^3k}{(2\pi)^3\,2\omega}\ \delta(K-p-q-k), \qquad (4.19)$$

where p_o, q_o and ω are the energies of π^+, π^- and γ respectively, K is the K meson four momentum and A is the amplitude $A(K_2^o\to\pi^+\pi^-\gamma)$.

$$A = 0.579\times10^{-7}\ \varepsilon_{\mu\nu\sigma\lambda}p^\mu q^\nu k^\sigma\varepsilon^\lambda[B] ,$$

with

$$B = [B_{st}+B_{su}+B_{tu}+(\xi-1)B_{tu}] .$$

We have shown in Appendix   that the function B does not vary appreciably over the Dalitz plot and hence can be regarded as constant. We shall take an average of B,

$$B = 11.88 + (\xi-1)\ 3.57\ , \qquad (4.20)$$

then

$$|A|^2 = (0.579\times10^{-7}B)^2[\varepsilon_{\mu\nu\sigma\lambda}p^\mu q^\nu k^\sigma \varepsilon^\lambda][\varepsilon_{\alpha\beta\gamma\delta}p^\alpha q^\beta k^\gamma \varepsilon^\delta]$$

$$= (0.579\times10^{-7}B)^2[2(p.q)(q.k)(p.k)-m_\pi^2(p.k)^2-m_\pi^2(q.k)^2],$$

where we have made use of the relations

$$\varepsilon^\lambda\varepsilon^\delta = \delta^{\lambda\delta},$$

$$\varepsilon_{\mu\nu\sigma\lambda}\varepsilon_{\alpha\beta\gamma\lambda} = g_{\mu\alpha}g_{\nu\beta}g_{\sigma\gamma} + g_{\mu\beta}g_{\nu\gamma}g_{\sigma\alpha} + g_{\mu\gamma}g_{\nu\alpha}g_{\sigma\beta}$$

$$- g_{\mu\beta}g_{\nu\alpha}g_{\sigma\gamma} - g_{\mu\gamma}g_{\sigma\alpha}g_{\nu\beta} - g_{\mu\alpha}g_{\nu\gamma}g_{\beta\sigma}.$$

We then get

$$\frac{1}{\tau} = \frac{1}{(2\pi)^5}\frac{1}{2K_o}\int[0.579\times10^{-7}B]^2\{2(p.q)(q.k)(p.k)-m_\pi^2(p.k)^2$$

$$- m_\pi^2(p.q)^2\}\ \frac{d^3pd^3qd^3k}{8p_oq_o\omega}\ \delta(K-p-q-k)\ , \qquad (4.21)$$

or

$$\frac{1}{\tau} = \frac{1}{(2\pi)^5}\frac{1}{2K_o}\int[0.579\times10^{-7}B]\{2(p.G-p.k)(G.k)(p.k)-m_\pi^2(p.k)^2$$

$$- m_\pi^2(G.k)^2\}\ \frac{2\pi|p|d\ \cos\theta\ d|p|}{2p_o}\ \frac{d^3k}{2\omega}\ \delta[G^2-2G.k-m_\pi^2]\ ,$$

where $G = K-p = k+q$. In the frame where K meson is at rest (i.e. $K = (m_K,0,0,0)$)

$$\delta[G^2 - 2G.k - m_\pi^2] = \frac{1}{2|p||k|}\,\delta\left[\cos\theta - \frac{(m_K-p_o-\omega)^2-p_o^2-\omega^2}{2|p||k|}\right]$$

and

$$\{2(p.G-p.k)(G.k)(p.k)-m_\pi^2(p.k)^2-m_\pi^2(G.k)^2\} = m_K^2|p|^2\omega^2\sin\theta \ ,$$

and the decay rate becomes,

$$\frac{1}{\tau} = \frac{1}{64\pi^3\, m_K}\int[0.579\times10^{-7}B]m_K^2|p|^2\omega^2\sin\theta\, d\,p_o d\omega d\,\cos\theta \ \times$$

$$\times\ \delta\left[\cos\theta - \frac{(m_K-p_o-\omega)^2-p_o^2-\omega^2}{2|p||k|}\right]$$

$$= \frac{m_K}{64\pi^3}\int_0^{\omega_{max}} d\omega \int_{p_o(\omega)_{min}}^{p_o(\omega)_{max}} dp_o[0.579\times10^{-7}B]^2\ \times$$

$$\times\ \{\omega^2(p_o^2-m_\pi^2) - \tfrac{1}{4}(m_K^2-2m_K p_o-2m_K\omega+2p_o\omega)^2\}$$

$$= [0.579\times10^{-7}B]^2\,\frac{1}{64\pi^3}\int_0^{\omega_{max}} d\omega\left\{\frac{m_K^2\omega^3}{3}\ \frac{(\omega_{max}-\omega)^{3/2}}{\left[(\omega_{max}-\omega)+\frac{2m_\pi^2}{m_K}\right]^{\frac{1}{2}}}\right\}$$

(4.22)

where

$$\omega_{max} = (m_K^2 - 4m_\pi^2)/2m_k\ ,$$

$$p_o(\omega)_{\substack{max\\min}} = \frac{1}{2}\left[m_K-\omega\pm\omega\sqrt{1-\frac{4m_\pi^2}{(m_K^2-2m_K\ \omega)}}\right] .$$

After the integration in (4.22) has been carried out, we obtain

$$\Gamma = \frac{1}{\tau} = [0.579\times10^{-7}B]^2 \frac{m_K^2}{64\pi^3} \frac{1}{3} \times 0.441 \times 10^{-5}$$

$$= \{4.73\times10^{-13}+2.84\times10^{-13}(\xi-1)+0.427\times10^{-13}(\xi-1)^2\} \times$$

$$\times \frac{(0.441 \times 10^{-5})m_K^2}{192\pi^3} ,$$

or

$$\Gamma = \{0.88\times10^{-22}+0.51\times10^{-22}(\xi-1)+0.08\times10^{-22}(\xi-1)^2\} \quad (\mathrm{Gev})$$

$$\Gamma = [1.3+0.75(\xi-1)+0.118(\xi-1)^2] \times 10^2 \qquad (\mathrm{sec}^{-1}) .$$

The branching ratio for $K_2^o\to\pi^+\pi^-\gamma$ is then given by

$$\frac{\Gamma(K_2^o\to\pi^+\pi^-\gamma)}{\Gamma(K_2^o\to\mathrm{all})} = [0.76+0.44(\xi-1)+0.068(\xi-1)^2] \times 10^{-5}. \quad (4.23)$$

If the off-mass shell effect of the intermediate pion in the strong interaction part is ignored then $\xi=1$ and the above branching ratio would be 0.76×10^{-5}. The experimental upper limit(23) for this branching ratio is 4×10^{-4}. The value of $|\xi|$ would have to be of order 10, if (4.23) were to be of the order 10^{-4}. Thus it seems that a simple model with essentially one parameter ξ can account for the observed branching ratio.

Section IV : Decay Rate for $K_2^0 \to \gamma\gamma$

Consider the $K_2^0 \to \gamma\gamma$ transition to proceed as shown in Fig. 4. The amplitude $A(K_2^0 \to \gamma\gamma)$ is written, in the pole model, as

$$A(K_2^0 \to \gamma\gamma) = \langle \pi'^0 | H_{wk} | K_2^0 \rangle \frac{2}{m_K^2 - m_{\pi'}^2} g_{\rho\pi'\gamma} \frac{e}{g_{\rho\pi\pi}} \varepsilon_{\mu\nu\sigma\lambda} \varepsilon_1^\mu k_1^\nu \varepsilon_2^\sigma k_2^\lambda \ , \tag{4.24}$$

where k_1 and k_2 are the four momenta of the photons and ε_1 and ε_2 are their polarizations, $g_{\rho\pi'\gamma}$ is assumed to be the same as $g_{\rho\pi\gamma}$ in which π is on the mass shell.

Using the value $\langle \pi'^0 | H_{wk} | K_2^0 \rangle / (m_K^2 - m_{\pi'}^2) = 0.5 \times 10^{-7}$ and $\Gamma(\rho \to \pi\gamma) \leq 0.5$ Mev(27), we get the amplitude for $K_2^0 \to \gamma\gamma$

$$A(K_2^0 \to \gamma\gamma) \leq \frac{1.83 \times 10^{-9}}{m_K} \varepsilon_{\mu\nu\sigma\lambda} \varepsilon_1^\mu k_1^\nu \varepsilon_2^\sigma k_2^\lambda \ , \tag{4.25}$$

and the branching ratio $= \dfrac{\Gamma(K_2^0 \to \gamma\gamma)}{\Gamma(K_2^0 \to \text{all})} \leq 7.1 \times 10^{-4}$.

If one uses the DESY result(28) $\Gamma(\rho \to \pi\gamma) \leq 0.24$ Mev, the result would be

$$A(K_2^0 \to \gamma\gamma) \leq \frac{1.3 \times 10^{-9}}{m_K} \varepsilon_{\mu\nu\sigma\lambda} \varepsilon_1^\mu k_1^\nu \varepsilon_2^\sigma k_2^\lambda \ , \tag{4.26}$$

and Branching ratio $= \dfrac{\Gamma(K_2^0 \to \gamma\gamma)}{K(K_2^0 \to \text{all})} \leq 3.6 \times 10^{-4}$.

We may use the vector dominance model and couple the second photon to an ω-meson (Fig. 5); then $g_{\rho\pi'\gamma}$ in (4.24) is replaced by$^{(22)}$ $eg_{\omega\pi'\rho}\sin\theta_Y/2f_Y$. Here, as in Section II, $g_{\omega\pi'\rho}$ is assumed to have the same value as $g_{\omega\pi\rho}$ in which π is on the mass shell. The amplitude now becomes

$$A(K_2^0\to\gamma\gamma) = \langle\pi'^0|H_{wk}|K_2^0\rangle \frac{2}{m_K^2-m_{\pi'}^2}\, eg_{\omega\pi'\rho}\, \frac{\sin\theta_Y}{2f_Y}\, \frac{e}{g_{\rho\pi\pi}} \times$$

$$\times\ \varepsilon_{\mu\nu\sigma\lambda}\varepsilon_1^\mu k_1^\nu \varepsilon_2^\sigma k_2^\lambda$$

$$= \frac{0.82\times10^{-9}}{m_K}\cdot\varepsilon_{\mu\nu\sigma\lambda}\varepsilon_1^\mu k_1^\nu \varepsilon_2^\sigma k_2^\lambda\ , \qquad (4.27)$$

which corresponds to a branching ratio 1.44×10^{-4}.

The experimental values for the $K_2^0\to\gamma\gamma$ branching ratio are 1.3×10^{-4} (29), 4.7×10^{-4} (30), 5.3×10^{-4} (31), 6.7×10^{-4} (26) and 7.4×10^{-4} (32). The result of (4.25) is consistent with the larger experimental value and the vector dominance model calculation (4.27) prefers the smaller value of the set of data. There are suggestions$^{(33)}$ that the correct experimental value might be close to 5×10^{-4}.

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APPENDIX

The Numerical Value of B in $K_2^0 \to \pi^+\pi^-\gamma$ over the Dalitz Plot

s (Gev^2)	t	u	B_{st}	B_{su}	B_{tu}	$B_{st}+B_{su}+B_{tu}$
s=0.08	t=0.105	u=0.105	3.81	3.81	4.01	11.63
0.097	0.097	0.097	3.86	3.86	3.86	11.58
0.11	0.09	0.09	3.86	3.86	3.7	11.42
0.12	0.085	0.085	3.91	3.91	3.7	11.50
0.13	0.08	0.08	3.94	3.94	3.64	11.52
0.16	0.065	0.065	4.08	4.08	3.5	11.66
0.2	0.05	0.05	4.24	4.24	3.36	11.84
0.25	0.02	0.02	4.56	4.56	3.13	12.25
0.097	0.13	0.063	4.05	3.66	3.88	11.59
0.1	0.12	0.07	3.99	3.66	3.77	11.42
0.13	0.1	0.06	4.05	3.77	3.64	11.46
0.13	0.12	0.04	4.7	3.76	3.7	11.63
0.2	0.06	0.03	4.25	4.19	3.31	11.75

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